HIGHER TRIGONOMETRY

OUR COLLEGE PUBLICATIONS

		Rs.	As.
1.	INTERMEDIATE ALGEBRA *(13th Edition)* By Ganguli & Mukherjee	. 5	0
2.	KEY TO INTERMEDIATE ALGEBRA	. 4	0
3.	INTERMEDIATE TRIGONOMETRY *(18th Edition)* By Das & Mukherjee	. 5	0
4.	KEY TO INTERMEDIATE TRIGONO-METRY	... 5	0
5.	KEY TO HIGHER TRIGONOMETRY	... 6	0
6.	INTEGRAL CALCULUS *(10th Edition)* By Das & Mukherjee	... 6	0
7.	KEY TO INTEGRAL CALCULUS	... 6	0
8.	INTERMEDIATE STATICS *(7th Edition)* By Das & Mukherjee	... 5	8
9.	KEY TO INTERMEDIATE STATICS	... 5	0
10.	INTERMEDIATE DYNAMICS *(7th Edition)* By Das & Mukherjee	... 5	8
11.	KEY TO INTERMEDIATE DYNAMICS	... 4	0
12.	CO-ORDINATE GEOMETRY *(6th Edition)* By Bagchi & Mukherjee	... 6	8
13.	CONIC SECTIONS, SOLID & CO-ORDINATE GEOMETRY *(4th Edition)* By Dhar & Ghose	3	0
14.	DIFFERENTIAL CALCULUS *(6th Edition)* By Das & Mukherjee	... 6	8
15.	KEY TO DIFFERENTIAL CALCULUS	... 6	0

Recommended by the Calcutta University as a text book for B.A. and B.Sc. Examinations (Pass and Honours)

HIGHER TRIGONOMETRY

BY

B. C. DAS, M. Sc.

PROFESSOR OF MATHEMATICS, PRESIDENCY COLLEGE, CALCUTTA (RETD.) :
LECTURER IN APPLIED MATHEMATICS, CALCUTTA UNIVERSITY

AND

B. N. MUKHERJEE, M. A.

Premchand Roychand Scholar

HEAD OF THE DEPARTMENT OF MATHEMATICS, SCOTTISH CHURCH COLLEGE, CALCUTTA

THIRTEENTH EDITION

U. N. DHUR & SONS, PRIVATE LTD.
15, BANKIM CHATTERJEE STREET,
CALCUTTA 12

Published by
UPENDRANATH DHUR,
For U. N. DHUR & SONS, PRIVATE LTD.,
15, Bankim Chatterjee St., Calcutta 12

THIRTEENTH EDITION—1931

Printed by
TRIDIBESH BASU, B.A.,
THE K. P. BASU PTG. WORKS,
11, Mohendra Gossain Lane, Calcutta 6

PREFACE

THIS book is prepared with a view to be used as a text-book for the B. A. and B. Sc. students (Pass and Honours) of the Indian Universities. We have tried to make the exposition clear and concise without going into unnecessary details. At the same time an attempt has been made to make the treatment rigorous and up-to-date.

The recent changes in the Trigonometry syllabus of the Calcutta University require an elementary notion of the convergence of the series used; so a chapter on the convergence of the series, particularly with reference to its application to Trigonometrical series, has been added as an Appendix.

Important formulæ and results of elementary Trigonometry, as also of this book, are given in the beginning for ready reference. A good number of typical examples has been worked out by way of illustrations.

Examples for exercise have been selected very carefully and include many, which have been set in the Pass and Honours examinations of different Universities. University questions of recent years have been added at the end to give the students an idea of the standard of the examinations.

Corrections and suggestions will be thankfully received.

B. C. D.
B. N. M.

PREFACE TO THE THIRD EDITION

THIS edition is practically a reprint of the second edition; only a few alterations have been made here and there. Our thanks are due to Prof. B. B. Mandal, M. Sc. of Scottish Church College, Prof. G. D. Dhar, M. Sc. of Presidency College and Prof. B. N. Pal, M. A. of St. Xavier's College and others, for their valuable criticisms and helpful suggestions in the revision of the text for the third edition.

B. C. D.
B. N. M.

PREFACE TO THE SIXTH EDITION

THIS edition differs very little from its previous edition; only an alternative method for finding the distance between the circum-centre and in-centre has been given at the end of Art. 36 and a new article on circum-radius, in-radius and area of a regular polygon has been added in the Chapter V. A few new types of examples have been added here and there especially in Chapters VIII and X, and the set of Miscellaneous Examples II has been shifted to the end of Chapter XII with some added examples.

B. C. D.
B. N. M.

CONTENTS

TRIGONOMETRY SYLLABUS

for B. A. and B. Sc. Examinations

of

THE CALCUTTA UNIVERSITY

Pass Course

1. Sub-multiple angles.
2. Properties of triangles.
3. Inverse circular functions.
4. Summation of finite series and infinite series.
5. Elementary notion of the convergence of series as applied to the exponential series, the logarithmic series and the sine series.
6. De Moivre's Theorem.
7. Exponential values of sine and cosine.
8. Expansions of sin θ and cos θ in powers of θ.

Honours Course

In addition to a fuller treatment of the Pass Course, the following :—

9. Expansion of $\sin^n\theta$, $\cos^n\theta$, $\sin n\theta$, $\cos n\theta$.
10. Hyperbolic functions.
11. Expansion in series.
12. Resolution of circular and hyperbolic functions into factors.

Abbreviations used in this book

1. C. P. means questions set in the B. A. and B. Sc. *Pass* Examinations of the *Calcutta* University.
2. C. H. means questions set in the B. A. and B. Sc. *Honours* Examinations of the *Calcutta* University.
3. P. P. means questions set in the B. A. and B. Sc. *Pass* Examinations of the *Patna* University.
4. C. M means questions set in the M. A. and M. Sc. Examinations of the *Calcutta* University.

IMPORTANT FORMULÆ AND RESULTS

FORMULÆ OF ELEMENTARY TRIGONOMETRY

I. A radian $= 57^\circ\ 17'\ 44{\cdot}8''$ nearly.

2 right angles $= 180^\circ = \pi$ radians.

$\pi = \frac{22}{7} = 3{\cdot}1416$ approximately.

Radian measure of an angle at the centre of a circle

$$= \frac{\text{subtending arc}}{\text{radius}}.$$

II. $\sin^2\theta + \cos^2\theta = 1$;

$\sec^2\theta = 1 + \tan^2\theta$;

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

III. $\sin 0^\circ = 0$; $\cos 0^\circ = 1$; $\tan 0^\circ = 0$.

$\sin 30^\circ = \frac{1}{2}$; $\cos 30^\circ = \frac{\sqrt{3}}{2}$; $\tan 30^\circ = \frac{1}{\sqrt{3}}$

$\sin 45^\circ = \frac{1}{\sqrt{2}}$; $\cos 45^\circ = \frac{1}{\sqrt{2}}$; $\tan 45^\circ = 1$.

$\sin 60^\circ = \frac{\sqrt{3}}{2}$; $\cos 60^\circ = \frac{1}{2}$; $\tan 60^\circ = \sqrt{3}$.

$\sin 90^\circ = 1$; $\cos 90^\circ = 0$; $\tan 90^\circ = \infty$.

$\sin 180^\circ = 0$; $\cos 180^\circ = -1$; $\tan 180^\circ = 0$.

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15^\circ = 2 - \sqrt{3}$.

$\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1)$.

IV. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; tah $(-\theta) = -\tan\theta$.

$\sin(90^\circ - \theta) = \cos\theta$; $\sin(90^\circ + \theta) = \cos\theta$.

$\cos(90° - \theta) = \sin\theta$; $\quad \cos(90° + \theta) = -\sin\theta$.

$\tan(90° - \theta) = \cot\theta$; $\quad \tan(90° + \theta) = -\cot\theta$.

$\sin(180° - \theta) = \sin\theta$; $\quad \sin(180° + \theta) = -\sin\theta$.

$\cos(180° - \theta) = -\cos\theta$; $\quad \cos(180° + \theta) = -\cos\theta$.

$\tan(180° - \theta) = -\tan\theta$; $\quad \tan(180° + \theta) = \tan\theta$.

sin *positive*	**all** *positive*
tan *positive*	**cos** *positive*

V. $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}.$$

VI. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

VII. $$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}.$$

VIII. $\sin 2A = 2 \sin A \cos A$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2\tan A}{1 + \tan^2 A}\,;\ \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

$$\left.\begin{aligned} 1 - \cos 2A &= 2\sin^2 A \\ 1 + \cos 2A &= 2\cos^2 A \end{aligned}\right\}$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

IX. $\sin 3A = 3\sin A - 4\sin^3 A$;

$$\cos 3A = 4\cos^3 A - 3\cos A\ ;$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

X. $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$;

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1$$

$$= 1 - 2\sin^2\frac{\theta}{2}.$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}\,;\ 1 + \cos\theta = 2\cos^2\frac{\theta}{2}.$$

XI. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$.

If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$.

If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$.

XII. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}.$$

XIII. $\log_a mn = \log_a m + \log_a n$

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

$$\log_a m^n = n \log_a m$$

$$\log_a m = \log_b m \times \log_a b$$

$$\log 1 = 0, \ \log_a a = 1.$$

XIV. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\left.\begin{aligned} \cos A &= \frac{b^2+c^2-a^2}{2bc} \\ \cos B &= \frac{c^2+a^2-b^2}{2ca} \\ \cos C &= \frac{a^2+b^2-c^2}{2ab} \end{aligned}\right\}$$

$$a = b\cos C + c\cos B.$$

$$b = c\cos A + a\cos C.$$

$$c = a\cos B + b\cos A.$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\triangle}{bc}$$

$$\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\triangle}{ca}$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\triangle}{ab}.$$

FORMULÆ OF HIGHER TRIGONOMETRY

XV. $\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$

$= \sqrt{s(s-a)(s-b)(s-c)},\quad 2s = a+b+c.$

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\triangle}$$

$$r = \frac{\triangle}{s} = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

$= (s-a)\tan\frac{1}{2}A = \text{etc.}$

$$r_1 = \frac{\triangle}{s-a} = 4R\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

$= s\tan\frac{1}{2}A.$

$$r_2 = \frac{\triangle}{s-b} = 4R\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

$= s\tan\frac{1}{2}B.$

$$r_3 = \frac{\triangle}{s-c} = 4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

$= s\tan\frac{1}{2}C.$

XVI. Area of a cyclic quadrilateral

$= \sqrt{(s-a)(s-b)(s-c)(s-d)},\quad 2s = a+b+c+d.$

Area of any quadrilateral

$= [(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha]^{\frac{1}{2}}$

where $\alpha = \frac{1}{2}(A+C)$.

XVII. $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta.$

XVIII. $\sin n\theta = {}^nc_1\cos^{n-1}\theta\sin\theta - {}^nc_3\cos^{n-3}\theta\sin^3\theta + \cdots$

$$\cos n\theta = \cos^n\theta - {}^nc_2 \cos^{n-2}\theta \sin^2\theta + \cdots$$

$$\tan n\theta = \frac{n \tan \theta - {}^nc_3 \tan^3\theta + \cdots}{1 - {}^nc_2 \tan^2\theta + {}^nc_4 \tan^4\theta - \cdots}$$

$$\tan (a_1 + a_2 + \cdots + a_n) = \frac{s_1 - s_3 + s_5 - \cdots}{1 - s_2 + s_4 - \cdots}$$

where s_r denotes the sum of the products of $\tan a_1$, $\tan a_2$,... $\tan a_n$, taken r at a time.

XIX. $\sin a = a - \dfrac{a^3}{3!} + \dfrac{a^5}{5!} - \cdots\cdots \infty$

$$\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \cdots\cdots \infty$$

$$\tan a = a + \tfrac{1}{3}a^3 + \tfrac{2}{15}a^5 + \cdots\cdots \infty$$

XX. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$$e^{ix} = \cos x + i \sin x \;;\; e^{-ix} = \cos x - i \sin x.$$

XXI. $\text{Log } (\alpha + i\beta) = \log_e \sqrt{\alpha^2 + \beta^2} + i(2n\pi + \theta)$,

where $\cos \theta = + \dfrac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$, $\sin \theta = + \dfrac{\beta}{\sqrt{\alpha^2 + \beta^2}}$.

XXII. $\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots \text{ to } \infty, \; -1 \leqslant x \leqslant +1.$

$$\theta = \tan \theta - \tfrac{1}{3} \tan^3\theta + \tfrac{1}{5} \tan^5\theta - \cdots \infty, \; -\tfrac{1}{4}\pi \leqslant \theta \leqslant +\tfrac{1}{4}\pi.$$

XXIII. $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \cdots$ to n terms

$$= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cdot \sin \left\{\alpha + \frac{n-1}{2}\beta\right\}$$

$$= \frac{\sin \frac{n \text{ diff}}{2}}{\sin \frac{\text{diff}}{2}} \cdot \sin \frac{\text{first angle} + \text{last angle}}{2}$$

$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \cdots$ to n terms

$$= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \left\{ \alpha + \frac{n-1}{2} \beta \right\}$$

$$= \frac{\sin \frac{n \text{ diff}}{2}}{\sin \frac{\text{diff}}{2}} \cdot \cos \frac{\text{first angle} + \text{last angle}}{2}.$$

XXIV. $\sinh x = \frac{1}{2}(e^x - e^{-x})$; $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$e^x = \cosh x + \sinh x$; $e^{-x} = \cosh x - \sinh x$

$\sin ix = i \sinh x$

$\cos ix = \cosh x$

$\tan ix = i \tanh x$

$\cosh^2 x - \sinh^2 x = 1$

$\operatorname{sech}^2 x + \tanh^2 x = 1$

$\coth^2 x - \operatorname{cosech}^2 x = 1.$

XXV. $x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$

$$= \prod_{r=0}^{n-1} \left\{ x^2 - 2ax \cos \left(\theta + \frac{2r\pi}{n} \right) + a^2 \right\}$$

$x^{2n} - 2x^n \cos n\theta + 1$

$$\prod_{r=0}^{n-1} \left\{ x^2 - 2x \cos \left(\theta + \frac{2r\pi}{n} \right) + 1 \right\}$$

$$x^n - 1 = (x^2 - 1) \prod_{r=1}^{\frac{1}{2}n-1} \left(x^2 - 2x \cos \frac{2r\pi}{n} + 1 \right) \quad (n \text{ even})$$

$$x^n - 1 = (x - 1) \prod_{r=1}^{\frac{n-1}{2}} \left(x^2 - 2x \cos \frac{2r\pi}{n} + 1 \right) \quad (n \text{ odd})$$

$$x^n + 1 = \prod_{r=1}^{\frac{n}{2}-1} \left(x^2 + 2x \cos \frac{2r+1}{n} \pi + 1 \right) \quad (n \text{ even})$$

$$x^n + 1 = (x + 1) \prod_{r=0}^{\frac{n-3}{2}} \left(x^2 - 2x \cos \frac{2r+1}{n} \pi + 1 \right) \quad (n \text{ odd}).$$

XXVI. $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots\cdots$ to ∞

$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots\cdots$ to ∞

XXVII. $\sinh\theta=\theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\cdots\cdots$ to ∞

$\cosh\theta=\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\left(1+\frac{4\theta^2}{5^2\pi^2}\right)\cdots\cdots$ to ∞

IMPORTANT RESULTS

1. (i) $\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n=e=2{\cdot}71828\ldots$

(ii) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots$ to ∞

(iii) $a^x=e^{x\log_e a}=1+x\log_e a+\frac{x^2}{2!}(\log_e a)^2+\cdots\cdots$ to ∞

(iv) $\log(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots\cdots$

(v) $\log(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\cdots\cdots$

(vi) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$

2. If $A+B+C=\pi$, then

(i) $\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$

(ii) $\cos A+\cos B+\cos C=1+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$

(iii) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(iv) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(v) $\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$

(vi) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$

(vii) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$

3. $\underset{\theta\to0}{Lt}\frac{\sin\theta}{\theta}=1$; $\underset{\theta\to0}{Lt}\cos\theta=1$; $\underset{\theta\to0}{Lt}\frac{\tan\theta}{\theta}=1.$

4. Area of a circle $=\pi r^2$.

Perimeter of a circle $=2\pi r$.

HIGHER TRIGONOMETRY

CHAPTER I

TRIGONOMETRICAL EQUATIONS AND GENERAL VALUES

1. It is known from elementary trigonometry that there are infinitely many angles, the trigonometrical ratios of which have a given value. Thus, the sine of each of the angles 30°, 150°, 390°, 510° etc. is equal to $\frac{1}{2}$. It is very convenient for the solution of trigonometrical equations as also for other purposes to obtain a general expression in a compact form embracing all angles the trigonometrical ratios of which have a given value.

2. General expression of all angles, one of whose trigonometrical ratios is zero.

If the sine of an angle be zero, from definition, the length of the perpendicular from any point on one of the arms upon the other is zero, so that the two arms must be in the same straight line. Evidently, therefore, such angles must be some multiple of π, odd or even.

Thus if $\mathbf{\sin\theta = 0}$, then $\mathbf{\theta = n\pi}$,

n being zero, or any integer, odd or even, positive or negative.

When the cosine of an angle is zero, the projection of any length along one arm upon the other is zero, and so the two arms must be at right angles to one another. The angle must therefore be evidently either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or differ

from these by complete revolutions; in other words, the angle may be any odd multiple of $\frac{\pi}{2}$.

Thus, if $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$,

n being zero, or any integer, odd or even, positive or negative.

Again, if $\tan\theta = 0$, then its numerator $\sin\theta$ is also zero,

and so $\theta = n\pi$.

Similarly, if $\cot\theta = 0$, then $\cos\theta = 0$,

and so $\theta = (2n+1)\frac{\pi}{2}$.

Note. Cosec θ or sec θ can never be zero, for they can never be numerically less than unity.

3. General expression of angles having the same sine *(or cosecant).*

Let α be any angle such that its sine is equal to a given quantity k numerically $\not> 1$; for fixing up the idea, and for the sake of convenience, in practice the smallest positive angle having for its sine the given quantity k is generally taken as α. Let θ be any other angle whose sine is equal to k.

Then, $\sin\theta = \sin\alpha$, or, $\sin\theta - \sin\alpha = 0$,

or, $2\sin\frac{1}{2}(\theta-\alpha)\cos\frac{1}{2}(\theta+\alpha) = 0$.

$\therefore$ either $\sin\frac{1}{2}(\theta-\alpha) = 0$,

i.e., $\frac{1}{2}(\theta-\alpha) =$ any multiple of $\pi = m\pi$ $\cdots$ (1)

or, else, $\cos\frac{1}{2}(\theta+\alpha) = 0$,

i.e., $\frac{1}{2}(\theta+\alpha) =$ any odd multiple of $\frac{\pi}{2} = (2m+1)\frac{\pi}{2}$ $\cdots$ (2)

From (1), $\theta-\alpha = 2m\pi$, *i.e.*, $\theta = \alpha + 2m\pi$. $\cdots$ (3)

From (2), $\theta+\alpha = (2m+1)\pi$, *i.e.*, $\theta = -\alpha + (2m+1)\pi$ $\cdots$ (4)

Combining (3) and (4), $\theta=(-1)^n\alpha+n\pi,$ $\cdots$ (5)
where n is zero, or any integer, positive or negative, odd or even.

If cosec $\theta=$ cosec α, then $\sin\theta=\sin\alpha$; hence all angles having the same cosecant as that of α are also given by the expression (5).

Thus, *all angles having the same sine or cosecant as that of α are given by*

$$\mathbf{n\pi+(-1)^n\alpha,}$$

n being zero, or any integer, positive or negative, odd or even.

4. General expression of angles having the same cosine *(or secant).*

Let α be the smallest positive angle such that its cosine is equal to a given quantity k numerically $\not> 1$; and let θ be any other angle whose cosine is equal to k.

Then, $\cos\theta=\cos\alpha$, or, $\cos\alpha-\cos\theta=0$.

$\therefore$ $2\sin\frac{1}{2}(\theta+\alpha)\sin\frac{1}{2}(\theta-\alpha)=0.$

$\therefore$ either $\sin\frac{1}{2}(\theta+\alpha)=0$,

i.e., $\frac{1}{2}(\theta+\alpha)=$ any multiple of $\pi=n\pi,$ $\cdots$ (1)

or, else, $\sin\frac{1}{2}(\theta-\alpha)=0$,

i.e., $\frac{1}{2}(\theta-\alpha)=$ any multiple of $\pi=n\pi$ $\cdots$ (2)

From (1), $\theta+\alpha=2n\pi,$ or, $\theta=2n\pi-\alpha.$ $\cdots$ (3)

From (2), $\theta-\alpha=2n\pi,$ or, $\theta=2n\pi+\alpha.$ $\cdots$ (4)

From (3) and (4), we have $\theta=2n\pi\pm\alpha,$ $\cdots$ (5)

where n is zero, or any integer, positive or negative.

It is also evident as in the previous case that all angles having the same secant as that of α are also included in the expression (5).

Hence, *all angles having the same cosine or secant as that of a are ginen by*

$$2n\pi \pm \alpha,$$

n being zero, or any integer, positive or negative, odd or even.

Note. As in Art. 3, instead of taking the smallest positive angle, we might take α to be any one angle having for its cosine the given quantity k. The general value of θ satisfying $\cos\theta = \cos\alpha$ as obtained above would not be affected at all.

5. General expression of all angles having the same tangent *(or cotangent).*

Let α be the smallest positive angle such that its tangent is equal to a given quantity k; and let θ be any other angle whose tangent is equal to k.

Then, $\tan\theta = \tan\alpha$, or, $\dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin\alpha}{\cos\alpha} = 0,$

or, $\dfrac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\theta\cos\alpha} = 0,$

or, $\dfrac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha} = 0.$ $\therefore$ $\sin(\theta-\alpha) = 0,$

i.e., $\theta - \alpha =$ any multiple of $\pi = n\pi$.

$\therefore$ $\theta = \alpha + n\pi.$ ... ... (1)

The factor $\dfrac{1}{\cos\theta\cos\alpha}$ cannot be zero, for cosine of an angle cannot have an infinitely large value.

It is also evident as in the previous cases that all angles having the same cotangent as that of α are given by the expression (1).

Hence, *all angles having the same tangent or cotangent as that of α are given by*

$$n\pi + \alpha,$$

n being zero, or any integer, positive or negative, odd or even.

Note. The remark below Art. 4 is applicable here also.

6. Special Cases.

From Art. 3, considering both the cases when n is odd or even, it may be easily seen that,

$$\text{if} \quad \sin\theta = 1 = \sin\frac{\pi}{2}, \; \theta = 2m\pi + \frac{\pi}{2} = (4m+1)\frac{\pi}{2},$$

$$\text{and if} \quad \sin\theta = -1 = \sin\left(-\frac{\pi}{2}\right), \; \theta = 2m\pi - \frac{\pi}{2} = (4m-1)\frac{\pi}{2},$$

$$\text{or,} \quad = (4k+3)\frac{\pi}{2},$$

where m, or $k(=m-1)$ is zero, or any integer, positive or negative.

Similarly, from Art. 4, it may be seen that

$$\text{if} \quad \cos\theta = 1, \qquad \theta = 2m\pi,$$

$$\text{and if} \quad \cos\theta = -1, \qquad \theta = (2m+1)\pi,$$

m being zero, or any integer, odd or even.

These are the usual forms in which the above special cases are used in practice.

7. Ex. 1. *Solve* $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$.

Multiplying both sides of the equation by $\frac{1}{\sqrt{2}}$, we have

$$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2}. \quad \therefore \cos\left(\theta + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}.$$

$$\therefore \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}. \qquad \therefore \theta = 2n\pi + \frac{\pi}{12}, \text{ or, } 2n\pi - \frac{7\pi}{12}.$$

Note. Extraneous solutions.

In general, the same trigonometrical equation may be solved by different methods, and the forms of result we arrive at, though apparently different in some cases, are ultimately equivalent. In some cases, however, we may solve a trigonometrical equation by methods leading to solutions which include in addition to the correct solutions, some extraneous solutions which do not satisfy the given

equation. The given equation which is of the type $a \cos \theta + b \sin \theta = c$ is an example. We proceed to demonstrate it as follows :

Here $\cos \theta - \dfrac{1}{\sqrt{2}} = \sin \theta.$

$\therefore \quad \cos^2\theta - \sqrt{2} \cos \theta + \frac{1}{2} = \sin^2\theta = 1 - \cos^2\theta,$
whence $2 \cos^2\theta - \sqrt{2} \cos \theta - \frac{1}{2} = 0.$

$\therefore \quad \cos \theta = \dfrac{\sqrt{2} \pm \sqrt{2+4}}{4} = \dfrac{1 \pm \sqrt{3}}{2\sqrt{2}} = \cos \dfrac{\pi}{12}, \text{ or, } \cos \dfrac{7\pi}{12}.$

$\therefore \quad \theta = 2n\pi \pm \frac{1}{12}\pi, \quad \text{or,} \quad 2n\pi \pm \frac{7}{12}\pi.$

But it can be easily seen on substitution that

$2n\pi - \frac{1}{12}\pi$ and $2n\pi + \frac{7}{12}\pi$ do not satisfy the given equation. The error in the method lies in squaring the equation as we have done, for the squared equation includes the equation $\cos \theta - \dfrac{1}{\sqrt{2}} = -\sin \theta$ *i.e.*, $\cos \theta + \sin \theta = \dfrac{1}{\sqrt{2}}$ of which the solutions are $2n\pi - \frac{1}{12}\pi$ and $2n\pi + \frac{7}{12}\pi$.

Equations of this type are therefore best solved as in Ex. 3 below, and not by squaring.

Thus, *while solving any trigonometrical equation, it is always advisable to verify the roots obtained*, for thereby extraneous roots, if any, can be easily detected.

Ex. 2. *Solve* $2 \sin^2 x + \sin^2 2x = 2.$ [*C. P. 1931, '37*]

The given equation can be written as

$$2(1 - \sin^2 x) - \sin^2 2x = 0, \quad \text{or,} \quad 2 \cos^2 x - 4 \sin^2 x \cos^2 x = 0,$$

or, $\quad 2 \cos^2 x\,(1 - 2 \sin^2 x) = 0, \quad$ or, $\quad \cos^2 x \cos 2x = 0.$

$\therefore \quad$ either, $\cos x = 0$, *i.e.*, $x = (2n+1)\dfrac{\pi}{2}$,

or, $\quad \cos 2x = 0$, *i.e.*, $2x = (2n+1)\dfrac{\pi}{2}$.

$\therefore \quad x = (2n+1)\dfrac{\pi}{4}.$

Ex. 3. *Solve $a \cos \theta + b \sin \theta + c = 0$, ($c \not> \sqrt{a^2+b^2}$).*

Put $a = r \cos \alpha$, $b = r \sin \alpha$, choosing the smallest positive value of α and keeping r positive.

Then, $r = \sqrt{a^2+b^2}$ and $\sin \alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos \alpha = \dfrac{a}{\sqrt{a^2+b^2}}$.

The signs of a and b will determine the quadrant in which α lies, and a and b being given, r and α are definitely known.

The equation now becomes, $r \cos (\theta - \alpha) = -c$,

$$\text{or,} \quad \cos (\theta - \alpha) = -\frac{c}{\sqrt{a^2+b^2}} = \cos \beta \text{ say,}$$

where β is the smallest angle whose cosine is $-\dfrac{c}{\sqrt{a^2+b^2}}$, and a, b, c being known, β is also known.

Then, $\theta - \alpha = 2n\pi \pm \beta$, or, $\theta = 2n\pi + \alpha \pm \beta$.

Note. Ex. 1 above is a particular case of this equation which is of a more general form.

Ex. 4. *If θ and ϕ satisfy the equation*

$$\sin \theta + \sin \phi = \sqrt{3} (\cos \phi - \cos \theta),$$

then will $\sin 3\theta + \sin 3\phi = 0$. [*C. P. 1925 ; C. H. 1924, '31.*]

Dividing both sides by 2 and transposing, we get

$$\frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta = \frac{\sqrt{3}}{2} \cos \phi - \frac{1}{2} \sin \phi.$$

$\therefore$ $\cos (\theta - \frac{1}{6}\pi) = \cos (\phi + \frac{1}{6}\pi)$.

$\therefore$ $\theta - \frac{1}{6}\pi = 2n\pi \pm (\phi + \frac{1}{6}\pi)$.

$\therefore$ $3\theta - \frac{1}{2}\pi = 6n\pi \pm 3(\phi + \frac{1}{6}\pi)$.

$\therefore$ $\cos (3\theta - \frac{1}{2}\pi) = \cos (3\phi + \frac{1}{2}\pi)$.

$\therefore$ $\sin 3\theta + \sin 3\phi = 0$.

Ex. 5. *If $\sin (\pi \cot x) = \cos (\pi \tan x)$, show that either cosec $2x$ or $\cot 2x$ is of the form $n + \frac{1}{4}$, where n is any integer, positive or negative.*

$$\sin (\pi \cot x) = \cos (\pi \tan x) = \sin (\tfrac{1}{2}\pi - \pi \tan x).$$

$$\therefore \quad \pi \cot x = k\pi + (-1)^k (\tfrac{1}{2}\pi - \pi \tan x) \; ;$$

when k is even, $= 2n$,

$\cot x + \tan x = 2n + \frac{1}{2}$

and when k is odd $= 2n + 1$,

$\cot x - \tan x = (2n+1) - \frac{1}{2} = 2n +$

$$\therefore \quad n+\tfrac{1}{2}=\tfrac{1}{2}(\cot x \pm \tan x)=\frac{1}{2}\left(\frac{\cos^2 x \pm \sin^2 x}{\sin x \cos x}\right)$$

$$=\frac{1}{\sin 2x}, \quad \text{or,} \quad \frac{\cos 2x}{\sin 2x}.$$

$\therefore$ either $\operatorname{cosec} 2x$, or, $\cot 2x = n+\tfrac{1}{2}$.

EXAMPLES I

1. Solve the following equations :

(i) (*a*) $1+\cos\theta = 2\cos^2\theta$. [*C. P. 1938.*]

(*b*) $3\sin\theta + 2\cos^2\theta = 0$. [*C. P. 1932.*]

(*c*) $\cos^3 x - \cos x \sin x - \sin^3 x = 1$.

(ii) $\tan^2 x + \cot^2 x = 2$.

(iii) $\tan\theta + \tan 2\theta = \tan 3\theta$. [*C. P. 1945.*]

(iv) $\cos x - \sin x = \sqrt{2}$. [*C. P. 1940.*]

(v) $4\sin 4x + 1 = \sqrt{5}$.

(vi) $\cos^3 x \sin 3x + \sin^3 x \cos 3x = \tfrac{3}{4}$.

(vii) $\cos 3x + \cos 2x + \cos x = 0$. [*C. P. 1939.*]

(viii) $\cos 3x + \sin 3x = \cos x + \sin x$.

(ix) $\tan x + \tan(x+\tfrac{1}{3}\pi) + \tan(x+\tfrac{2}{3}\pi) = 3$.

(x) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$.

(xi) $\sin m\theta + \sin n\theta = 0$. [*C. P. 1935.*]

(xii) $\sin m\theta + \cos n\theta = 0$.

(xiii) $\sin 2\theta \sec 4\theta + \cos 2\theta = \cos 6\theta$. [*C. P. 1934.*]

(xiv) $\sec(\tfrac{1}{4}\pi + x) + \sec(\tfrac{1}{4}\pi - x) = 2\sqrt{2}$.

(xv) $\operatorname{cosec} 4a - \operatorname{cosec} 4x = \cot 4a - \cot 4x$.

(xvi) $2(\sin 2x + \sin 2y) - 1 = 2\sin(x+y)$.

(xvii) $2\sin(\theta-\phi) = \sin(\theta+\phi) = 1$. [*C. H. 1937.*]

2. Find all the angles between 0 and 2π which satisfy the equations

(i) $4\cos^2\theta - 2\sqrt{2}\cos\theta - 1 = 0$. [*C. P. 1930.*]

(ii) $\cos 2\theta - \cos\theta + \frac{1}{2} = 0$.

3. Find the roots common to the equations

$$2\sin^2 x + \sin^2 2x = 2$$

and $$\sin 2x + \cos 2x = \tan x.$$

4. If $\sin x + \sin^2 x = 1$, find $\sin x$: and show that

$$\cos^2 x + \cot^4 x = 1.$$

5. Verify that the trigonometrical equation,

$$\begin{vmatrix} 1 & \cos\theta & \cos^2\theta \\ 1 & \cos\alpha & \cos^2\alpha \\ 1 & -\cos\alpha & \cos^2\alpha \end{vmatrix} = 0$$

leads to the final solution $\theta = n\pi \pm \alpha$, where n is zero or an integer.

6. Solve :

(i) $\sin\frac{1}{2}(n+1)\theta + \sin\frac{1}{2}(n-1)\theta = \sin\theta$. [*C. P. 1936.*]

(ii) $\tan x \cot(x+\alpha) = \tan\beta\cot(\beta+\alpha)$.

(iii) $\cos(x-a)\cos(x-b) = \sin a \sin b + \cos x \cos c$.

(iv) $\cos(x-a)\cos(x-b)\cos(x-c)$

$= \sin a \sin b \sin c \sin x + \cos a \cos b \cos c \cos x$.

[*Expand the cosines and divide by* $\cos a \cos b \cos c \cos^3 x$.]

7. Solve for x and y the equations

$$\sin x + \sin y + \sin a = \cos x + \cos y + \cos a,$$

$$x + y = 2a.$$

8. If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation

$x^3 - (a+1)x^2 + (b-a)x - b = 0$, prove that

$\alpha + \beta + \gamma = n\pi + \frac{1}{4}\pi$. [*C. H. 1939.*]

9. If x_1, x_2, x_3, x_4 are the roots of

$$\tan(x+\tfrac{1}{4}\pi)-3\tan 3x$$

no two of which have equal tangents, then

$$\tan x_1+\tan x_2+\tan x_3+\tan x_4=0.$$

[*Express the equation as a biquadratic in tan x.*]

10. Solve for x the equation,

$$\begin{vmatrix} 1 & \cos x & 0 & 0 \\ \cos x & 1 & \cos\alpha & \cos\beta \\ 0 & \cos\alpha & 1 & \cos\gamma \\ 0 & \cos\beta & \cos\gamma & 1 \end{vmatrix}=0.$$

[*Multiplying the first column by cos x and subtracting it from the second, reduce the order of determinant by one and then proceed.*]

Solve *(Ex. 11 to 14)* :

11. (i) $\cos^2 x\cos^2 y+\sin^2 x\sin^2 y=1.$

(ii) $\cot x-\cot y=2,\ 2\sin x\sin y=1.$

12. $\cos x+\cos y=\cos z,$

$\cos 2x+\cos 2y=\cos 2z,$

$\cos 3x+\cos 3y=\cos 3z.$

13. $\cos 2x+\cos 2(x-a)+\cos 2(x-b)+\cos 2(x-c)$

$$=4\cos a\cos b\cos c.$$

[$4\cos a\cos b\cos c=\cos(a+b+c)+\cos(-a+b+c)$
$+\cos(a-b+c)+\cos(a+b-c).$]

14. $\left(\dfrac{\sin\theta}{\sin\phi}\right)^2=\dfrac{\tan\theta}{\tan\phi}=3.$ [*C. P. 1942.*]

15. Show that the equation $\sin x(\sin x+\cos x)=a$ has real solution if and only if a is a real number lying between

$$\tfrac{1}{2}(1-\sqrt{2}) \text{ and } \tfrac{1}{2}(1+\sqrt{2}).$$

ANSWERS

1. (i) (*a*) $2n\pi$ or $2n\pi \pm \frac{2}{3}\pi$. (*b*) $n\pi + (-1)^n \frac{7\pi}{6}$. (*c*) $2n\pi$ or $2n\pi - \frac{\pi}{2}$.

(ii) $n\pi \pm \frac{\pi}{4}$. (iii) $n\pi$, $\frac{n\pi}{3}$ or $\frac{n\pi}{2}$. (iv) $2n\pi - \frac{\pi}{4}$. (v) $\frac{n\pi}{4} + (-1)^n \frac{\pi}{40}$.

(vi) $\frac{n\pi}{4} + (-1)^n \frac{\pi}{8}$. (vii) $2n\pi \pm \frac{2\pi}{3}$ or $n\pi \pm \frac{\pi}{4}$.

(viii) $n\pi$ or $\frac{n\pi}{2} + \frac{\pi}{8}$. (ix) $\frac{\pi}{3}\left(n + \frac{1}{4}\right)$.

(x) $2n\pi \pm \frac{\pi}{2}$, $4n\pi \pm \pi$, $\frac{2}{5} n\pi$; the first two answers can be put in the forms $(2k+1)\frac{\pi}{2}$ and $(2k+1)\pi$. (xi) $\frac{k\pi}{m + (-1)^k n}$. (xii) $\frac{4k-1}{m \pm n} \cdot \frac{\pi}{2}$.

(xiii) $\theta = \frac{1}{2} n\pi$ or $\left(4n - 1\right) \frac{\pi}{16}$. (xiv) $\frac{2}{3} n\pi$. (xv) $\frac{1}{2} n\pi + a$.

(xvi) $x + y = n\pi + (-1)^n \frac{\pi}{6}$ and $x - y = 2m\pi \pm \frac{\pi}{3}$.

(xvii) $$\left.\begin{aligned} \theta &= r\pi + \frac{\pi}{3} \\ \phi &= (2n - r)\pi + \frac{\pi}{6} \end{aligned}\right\} \text{ or } \begin{aligned} \theta &= r\pi + \frac{2\pi}{3} \\ \phi &= (2n - r)\pi - \frac{\pi}{6}. \end{aligned}$$

2. (i) $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$. (ii) $\frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$.

3. $n\pi + \frac{\pi}{4}$ and $n\pi + \frac{3\pi}{4}$; they can also be put in the form $(2k+1)\frac{\pi}{4}$.

4. $\frac{\sqrt{5} - 1}{2}$. **6.** (i) $(2k+1)\pi$ or $\frac{2k\pi}{n + (-1)^{k+1}}$. (ii) $n\pi + \beta$, $n\pi + \frac{\pi}{2} - a - \beta$.

(iii) $n\pi + \frac{1}{2}\pi$; $2n\pi + a + b \pm c$. (iv) $n\pi$ or $n\pi + (a + b + c)$.

7. $x = a + 2n\pi \pm \frac{2\pi}{3}$; $y = a - 2n\pi \mp \frac{2\pi}{3}$.

10. $n\pi + (-1)^n . \sin^{-1} \frac{\sqrt{\cos^2 a + \cos^2 \beta - 2 \cos a \cos \beta \cos \gamma}}{\sin \gamma}$.

11. (i) $x = m\pi$, $y = n\pi$; $x = m\pi + \frac{1}{3}\pi$, $y = n\pi + \frac{1}{2}\pi$.

(ii) $x = (m - n)\pi + \frac{1}{4}\pi$, $y = (m + n)\pi + \frac{3}{4}\pi$.

12. $x = 2n\pi \pm \frac{1}{3}\pi$, $y = 2m\pi \pm \frac{2}{3}\pi$ or $x = 2n\pi \pm \frac{2}{3}\pi$, $y = 2m\pi \pm \frac{1}{3}\pi$; and $s = k\pi + \frac{1}{2}\pi$.

13. $x = n\pi + \frac{1}{2}(a + b + c)$ or $n\pi - \frac{1}{2}(a + b + c) + \tan^{-1} \frac{\Sigma \sin 2a}{1 + \Sigma \cos 2a}$.

14. $\theta = m\pi \pm \frac{1}{3}\pi$, $\phi = n\pi \pm \frac{1}{6}\pi$ (only like signs being paired).

CHAPTER II

SUBMULTIPLE ANGLES

8. From the usual formulæ for multiple angles, namely

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$1 + \cos 2A = 2 \cos^2 A,\ 1 - \cos 2A = 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A},$$

putting $A = \frac{1}{2}\theta$ and $\frac{1}{3}\theta$ respectively, we derive the following formulæ for submultiple angles.

$$\sin \theta = 2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta$$

$$\cos \theta = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta = 2 \cos^2 \tfrac{1}{2}\theta - 1 = 1 - 2 \sin^2 \tfrac{1}{2}\theta$$

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2},\ 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}$$

$$\sin \theta = 3 \sin \tfrac{1}{3}\theta - 4 \sin^3 \tfrac{1}{3}\theta$$

$$\cos \theta = 4 \cos^3 \tfrac{1}{3}\theta - 3 \cos \tfrac{1}{3}\theta$$

$$\tan \theta = \frac{3 \tan \frac{1}{3}\theta - \tan^3 \frac{1}{3}\theta}{1 - 3 \tan^2 \frac{1}{3}\theta}.$$

9. Values of sin $\frac{1}{2}\theta$ and cos $\frac{1}{2}\theta$ in terms of cos θ.

From $1-\cos\theta=2\sin^2\frac{\theta}{2}$ and $1+\cos\theta=2\cos^2\frac{\theta}{2}$, we at once deduce

$$\sin\tfrac{1}{2}\theta=\pm\sqrt{\tfrac{1}{2}(1-\cos\theta)}$$
$$\cos\tfrac{1}{2}\theta=\pm\sqrt{\tfrac{1}{2}(1+\cos\theta)}.$$

10. *Ambiguity of signs explained.*

When $\cos\theta$ is given and not θ, θ and consequently $\frac{1}{2}\theta$ has a series of values. If α be the smallest positive value of θ for the given value of $\cos\theta$, the general value of θ is $2n\pi\pm\alpha$, n being any integer. Thus for different values of n, $\frac{1}{2}\theta$ may lie in any of two possible quadrants depending on the value of α, and $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ will then have corresponding signs.

If the quadrant in which $\frac{1}{2}\theta$ lies be known, for example when θ is given along with $\cos\theta$, there is no ambiguity in choosing the proper signs of $\cos\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta$, as shown in the following example.

Ex. 1. *Find* $\sin 22\frac{1}{2}°$ *and* $\cos 22\frac{1}{2}°$.

$$\sin 22\tfrac{1}{2}°=+\sqrt{\tfrac{1}{2}(1-\cos 45°)}=\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)}=\frac{1}{2}\sqrt{2-\sqrt{2}};$$

$$\cos 22\tfrac{1}{2}°=+\sqrt{\tfrac{1}{2}(1+\cos 45°)}=\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)}=\frac{1}{2}\sqrt{2+\sqrt{2}}.$$

11. Values of sin $\frac{1}{2}\theta$ and cos $\frac{1}{2}\theta$ in terms of sin θ.

We know that $\sin\theta=2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

and $\quad 1=\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta.$

Therefore, $\quad 1+\sin\theta=(\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta)^2$

and $\quad 1-\sin\theta=(\cos\frac{1}{2}\theta-\sin\frac{1}{2}\theta)^2.$

Hence, $\quad \cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta=\pm\sqrt{1+\sin\theta}$

$\quad \cos\frac{1}{2}\theta-\sin\frac{1}{2}\theta=\pm\sqrt{1-\sin\theta}.$

Thus, $$\cos \tfrac{1}{2}\theta = \pm \tfrac{1}{2}\sqrt{1+\sin\theta} \pm \tfrac{1}{2}\sqrt{1-\sin\theta}$$

and $$\sin \tfrac{1}{2}\theta = \pm \tfrac{1}{2}\sqrt{1+\sin\theta} \mp \tfrac{1}{2}\sqrt{1-\sin\theta}.$$

12. *Ambiguity of signs explained.*

As before, when $\sin\theta$ is given and not θ, θ has a series of values given by $n\pi+(-1)^n\alpha$, where α is the smallest positive value of θ for the given value of $\sin\theta$; $\frac{1}{2}\theta$ will therefore have values lying in different quadrants, $\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta = \sqrt{2}\sin(\frac{1}{4}\pi+\frac{1}{2}\theta)$ and $\cos\frac{1}{2}\theta-\sin\frac{1}{2}\theta = \sqrt{2}\cos(\frac{1}{4}\pi+\frac{1}{2}\theta)$ will have their signs determined accordingly.

If in addition to the given value of $\sin\theta$, of the four quadrants determined by $2n\pi\pm\frac{1}{4}\pi$, $(2n+\frac{1}{2})\pi\pm\frac{1}{4}\pi$, $(2n+1)\pi\pm\frac{1}{4}\pi$ and $(2n+\frac{3}{2})\pi\pm\frac{1}{4}\pi$, thatin which $\frac{1}{2}\theta$ specifically lies be given, there is no ambiguity in choosing the signs of $\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta-\sin\frac{1}{2}\theta$. Thus, $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ will be definitely known.

Ex. 2. *Find $\sin 15°$ and $\cos 15°$.*

We have, $\cos 15°+\sin 15° = +\sqrt{1+\sin 30°} = \sqrt{1+\frac{1}{2}}$

$$\cos 15°-\sin 15° = +\sqrt{1-\sin 30°} = \sqrt{1-\tfrac{1}{2}}.$$

[$\cos 15°-\sin 15° = \sqrt{2}\sin(\frac{1}{4}\pi-15°)$ and is clearly positive]

Thus, $$\cos 15° = \tfrac{1}{2}\left(\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{1}{2}}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\sin 15° = \tfrac{1}{2}\left(\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

13. $\sin \frac{1}{2^n}\theta$ and $\cos \frac{1}{2^n}\theta$ in terms of $\cos\theta$.

We have, $2\cos\frac{1}{2}\theta = 2\sqrt{\frac{1}{2}(1+\cos\theta)} = \sqrt{2+2\cos\theta}$.

Similarly, $2\cos\frac{1}{2^2}\theta = \sqrt{2+2\cos\frac{1}{2}\theta} = \sqrt{2+\sqrt{2+2\cos\theta}}$.

$$2\cos\frac{1}{2^3}\theta = \sqrt{2+2\cos\frac{1}{2^2}\theta} = \sqrt{2+\sqrt{2+\sqrt{2+2\cos\theta}}},$$

and so on.

In the same way,

$$2\cos\frac{1}{2^n}\theta=\sqrt{2+2\cos\frac{1}{2^{n-1}}\theta}$$

$$=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos\theta}}}}$$

n square root signs being involved.

$$2\sin\frac{1}{2^n}\theta=2\sqrt{\frac{1}{2}\left(1-\cos\frac{1}{2^{n-1}}\theta\right)}$$

$$=\sqrt{2-2\cos\frac{1}{2^{n-1}}\theta}$$

$$=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos\theta}}}}}$$

n square root signs being involved.

Cor. $\cos\frac{1}{8}\pi=\frac{1}{2}\sqrt{2+\sqrt{2}}$; $\cos\frac{1}{16}\pi=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

$\cos\frac{1}{32}\pi=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$; etc.

$\sin\frac{1}{8}\pi=\frac{1}{2}\sqrt{2-\sqrt{2}}$; $\sin\frac{1}{16}\pi=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$

$\sin\frac{1}{32}\pi=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$; etc.

14. tan $\frac{1}{2}\theta$ in terms of tan θ.

From the formula, $\tan\theta=\dfrac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$

i.e., from the $\tan\theta.\tan^2\frac{1}{2}\theta+2\tan\frac{1}{2}\theta-\tan\theta=0$,

we easily dedue

$$\tan\frac{1}{2}\theta=\frac{-1\pm\sqrt{1+\tan^2\theta}}{\tan\theta}.$$

The reason of the ambiguity is similar to those in the previous cases.

15. Ratios of $\frac{1}{3}\theta$ from those of θ.

By solving the cubic equation

$$\sin\theta = 3\sin\tfrac{1}{3}\theta - 4\sin^3\tfrac{1}{3}\theta \qquad \cdots \quad (1)$$

we get $\sin\frac{1}{3}\theta$, if $\sin\theta$ be known.

Now, a cubic equation has three roots; we therefore get three possible values of $\sin\frac{1}{3}\theta$. The reason may be explained as follows.

All angles, for which $\sin\theta$ is given, are included in the general formula $n\pi + (-1)^n\theta$, n being any integer *i.e.*, $2m\pi + \theta$ and $(2m+1)\pi - \theta$, where m is any integer. The values of $\sin\frac{1}{3}\theta$ derived from (1) therefore, will include values of $\sin\frac{1}{3}(2m\pi + \theta)$ and $\sin\frac{1}{3}\{(2m+1)\pi - \theta\}$ for all integral values of m. Now m, when divided by 3, will leave remainders 0, 1 or 2. As any multiple of 2π, added or subtracted, will not alter the sine of an angle, $\sin\frac{1}{3}(2m\pi + \theta)$ will have three distinct values, *viz.*, $\sin\frac{1}{3}\theta$, $\sin\frac{1}{3}(2\pi + \theta)$, $\sin\frac{1}{3}(4\pi + \theta)$. Similarly, $\sin\frac{1}{3}\{(2m+1)\pi - \theta\}$ will have three distinct values, *viz.*, $\sin\frac{1}{3}(\pi - \theta)$, $\sin\frac{1}{3}(3\pi - \theta)$, $\sin\frac{1}{3}(5\pi - \theta)$. Now, $\sin\frac{1}{3}(\pi - \theta) = \sin\{\pi - \frac{1}{3}(\pi - \theta)\} = \sin\frac{1}{3}(2\pi + \theta)$; $\sin\frac{1}{3}(3\pi - \theta) = \sin\frac{1}{3}\theta$; $\sin\frac{1}{3}(5\pi - \theta) = \sin\{3\pi - \frac{1}{3}(4\pi + \theta)\} = \sin\frac{1}{3}(4\pi + \theta)$.

Thus, $\sin\frac{1}{3}\{n\pi + (-1)^n\theta\}$ for different integral values of n has only three distinct values, *viz.*, $\sin\frac{1}{3}\theta$, $\sin\frac{1}{3}(2\pi + \theta)$, $\sin\frac{1}{3}(4\pi + \theta)$ which are obtained as solutions of (1).

Obs. The choice of the particular value from these three depends on the circumstances of any particular case and on the relative magnitudes and signs of the three values. As an example, if we want to find $\sin 10^\circ$ from $\sin 30^\circ$, as the three roots of the corresponding equation, we shall get values of $\sin 10^\circ$, $\sin 130^\circ$, $\sin 250^\circ$. Now $\sin 250^\circ$ is negative and $\sin 130^\circ$ ($= \sin 50^\circ$), though positive, is greater than $\sin 10^\circ$. Thus, the least positive solution is to be selected in this case.

Solving, as above, the cubic equations

$$\cos\theta = 4\cos^3\tfrac{1}{3}\theta - 3\cos\tfrac{1}{3}\theta \quad \cdots \quad \cdots (2)$$

and
$$\tan\theta = \frac{3\tan\frac{1}{3}\theta - \tan^3\frac{1}{3}\theta}{1-3\tan^2\frac{1}{3}\theta} \quad \cdots \quad \cdots (3)$$

we derive values of $\cos\frac{1}{3}\theta$ from that of $\cos\theta$, and of $\tan\frac{1}{3}\theta$ from that of $\tan\theta$ respectively.

Discussing as before, it may be shown that the three roots of (2) are the values of $\cos\frac{1}{3}\theta$, $\cos\frac{1}{3}(2\pi+\theta)$, $\cos\frac{1}{3}(4\pi+\theta)$ and those of (3) are the values of $\tan\frac{1}{3}\theta$, $\tan\frac{1}{3}(\pi+\theta)$, $\tan\frac{1}{3}(2\pi+\theta)$.

16. Ratios of 18° and 36°.

Let $\theta = 18°$; then $2\theta = 90° - 3\theta$.

$\therefore \quad \sin 2\theta = \cos 3\theta$, or $2\sin\theta\cos\theta = \cos\theta(4\cos^2\theta - 3)$.

As $\cos\theta$ (*i.e.*, $\cos 18°$) is not zero, we have

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta,$$

or, $\quad 4\sin^2\theta + 2\sin\theta - 1 = 0.$

$$\therefore \quad \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{(\pm\sqrt{5}-1)}{4}.$$

Now, as θ here is a positive acute angle, therefore rejecting the negative value, we get

$$\mathbf{\sin 18° = \tfrac{1}{4}(\sqrt{5}-1).}$$

$$\mathbf{\cos 18° = +\sqrt{1-\sin^2 18°} = \tfrac{1}{4}(\sqrt{10+2\sqrt{5}}).}$$

$$\mathbf{\cos 36° = 1 - 2\sin^2 18° = \tfrac{1}{4}(\sqrt{5}+1).}$$

$$\mathbf{\sin 36° = \sqrt{1-\cos^2 36°} = \tfrac{1}{4}(\sqrt{10-2\sqrt{5}}).}$$

Note. Since 54° and 36° are complementary and 72° and 18° are complementary, from the above values we easily get the trigonometrical ratios of 54° and 72°.

17. Ratios of 3° and multiples of 3°.

$\sin 3^\circ = \sin(18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ$
$= \frac{1}{16}(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \frac{1}{8}(\sqrt{3}-1)(\sqrt{5+\sqrt{5}})$

on substituting the values of sin 18°, cos 15° etc.

Similarly,

$$\cos 3^\circ = \frac{1}{8}(\sqrt{3}+1)(\sqrt{5+\sqrt{5}}) + \frac{1}{16}(\sqrt{6}-\sqrt{2})(\sqrt{5}-1).$$

From a knowledge of the ratios of 3°, 15°, 18°, 30°, 36° and 45°, we can deduce the ratios for all angles which are multiples of 3°, (for, $6^\circ = 36^\circ - 30^\circ$; $9^\circ = 45^\circ - 36^\circ$; $12^\circ = 30^\circ - 18^\circ$; $21^\circ = 36^\circ - 15^\circ$; etc.). For angles greater than 45°, the ratios may be deduced from those of their complements which are less than 45°.

Ex. 3. *Prove that*

$$\tan a \tan(\tfrac{1}{3}\pi + a) \tan(\tfrac{2}{3}\pi + a) = -\tan 3a.$$

In the relation $\tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$, putting $\tan a = x$,

we get, $\tan 3a = \frac{3x - x^3}{1 - 3x^2}$,

or, $x^3 - 3x^2 \tan 3a - 3x + \tan 3a = 0$.

Let x_1, x_2, x_3 be the three roots of this cubic equation. It at once follows from Art. 15 that $x_1 = \tan a$, $x_2 = \tan(\frac{1}{3}\pi + a)$, $x_3 = \tan(\frac{2}{3}\pi + a)$. Also, it is known from the Theory of Equations that $x_1x_2x_3 = -\tan 3a$; hence the required result follows.

Ex. 4. *Show that*

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cdots$$

We have, $\sin x = 2\sin\frac{1}{2}x \cos\frac{1}{2}x$;

$$\sin\frac{1}{2}x = 2\sin\frac{1}{2^2}x\cos\frac{1}{2^2}x;$$

$$\sin\frac{1}{2^2}x = 2\sin\frac{1}{2^3}x\cos\frac{1}{2^3}x.$$

Similarly, $\sin \frac{1}{2^{n-1}} x = 2 \sin \frac{1}{2^n} x \cos \frac{1}{2^n} x$.

Hence, $\sin x = 2^n \cos \frac{1}{2} x \cos \frac{1}{2^2} x \cos \frac{1}{2^3} x \cdots\cdots \cos \frac{1}{2^n} x \sin \frac{1}{2^n} x$.

When n is indefinitely large, $\frac{1}{2^n} x$ is indefinitely small and hence in the limit, $\sin \frac{1}{2^n} x = \frac{1}{2^n} x$.

Hence,
$$\frac{\sin x}{x} = \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cdots\cdots$$

Note. This is known as *Euler's Product*.

EXAMPLES II

1. Determine the limits between which A must lie in order that

(i) $2 \sin A = \sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}$.

(ii) $2 \cos A = -\sqrt{1 + \sin 2A} + \sqrt{1 - \sin 2A}$.

2. Prove that

(i) $\tan 7\frac{1}{2}^\circ = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

(ii) $\tan 142\frac{1}{2}^\circ = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$.

3. Prove that
$$\frac{\cos \frac{1}{2}A}{\sqrt{1 + \sin A}} + \frac{\sin \frac{1}{2}A}{\sqrt{1 - \sin A}} = \sec A,$$
provided A lies between $(4n - \frac{1}{2})\pi$ and $(4n + \frac{1}{2})\pi$, the positive sign of the radicals being taken.

4. If $A = 240^\circ$, is the following statement co
$$2 \sin \tfrac{1}{2}A = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}.$$
If not, how must it be modified ?

5. Prove that the four values of $\dfrac{\sqrt{1-\sin x}+1}{\sqrt{1+\sin x}-1}$ are

$$\cot \tfrac{1}{4}x,\ \tan \tfrac{1}{4}(\pi+x),\ -\tan \tfrac{1}{4}x,\ -\cot \tfrac{1}{4}(\pi+x).$$

6. If $\tan \alpha$ be given and $\tan \frac{1}{2}\alpha$ be found in terms of it, prove that there will be two values of $\tan \frac{1}{2}\alpha$ reciprocals in magnitude and opposite in sign.

7. If $\tan \dfrac{\theta}{2} = \sqrt{\dfrac{1-e}{1+e}} \tan \dfrac{\phi}{2}$, show that

$$\cos \phi = \frac{\cos \theta - e}{1 - e \cos \theta}.$$

8. Prove that

(i) $\cos^4 \dfrac{\pi}{8} + \cos^4 \dfrac{3\pi}{8} + \cos^4 \dfrac{5\pi}{8} + \cos^4 \dfrac{7\pi}{8} = \dfrac{3}{2}$.

(ii) $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$.

(iii) $\sin \dfrac{\pi}{16} \sin \dfrac{3\pi}{16} \sin \dfrac{5\pi}{16} \sin \dfrac{7\pi}{16} = \dfrac{1}{8\sqrt{2}}$.

(iv) $\tan \alpha + \tan (\frac{1}{3}\pi + \alpha) + \tan (\frac{2}{3}\pi + \alpha) = 3 \tan 3\alpha$.

9. Prove that one root of the equation

$$8x^4 + 4x^3 - 6x^2 - 2x + \tfrac{1}{2} = 0 \text{ is } \sin \frac{\pi}{18}.$$

10. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, find the value of $\tan \frac{1}{2}(\alpha - \beta)$. [*C. P. 1940.*]

11. If $\alpha + \beta + \gamma = \frac{1}{2}\pi$, show that

$$\frac{(1-\tan \frac{1}{2}\alpha)(1-\tan \frac{1}{2}\beta)(1-\tan \frac{1}{2}\gamma)}{(1+\tan \frac{1}{2}\alpha)(1+\tan \frac{1}{2}\beta)(1+\tan \frac{1}{2}\gamma)}$$

$$= \frac{\sin \alpha + \sin \beta + \sin \gamma - 1}{\cos \alpha + \cos \beta + \cos \gamma}.$$

12. If $\theta = \dfrac{1}{2^n+1}\pi$, prove that

$$2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = 1.$$

13. Show that

(i) $\dfrac{\theta}{\tan\theta} = \left(1-\tan^2\dfrac{\theta}{2}\right)\left(1-\tan^2\dfrac{\theta}{2^2}\right)\left(1-\tan^2\dfrac{\theta}{2^3}\right)\ldots\ldots$

[*P. H. 1935.*]

(ii) $\dfrac{\tan 2^n\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\cdots$

$\cdots(1+\sec 2^n\theta).$

[*Use* $\tan\theta\,(1+\sec 2\theta) = \tan 2\theta$.]

(iii) $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)$

$\times(2\cos 2^2\theta-1)\ldots\ldots(2\cos 2^{n-1}\theta-1).$

[*Use* $(2\cos\theta+1)(2\cos\theta-1) = 2\cos 2\theta+1$.]

14. Show that

$$\frac{2}{\pi}^* = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots\cdots$$

[*Put* $x = \frac{1}{2}\pi$ *in Euler's product and use Art. 13.*]

ANSWERS

1. (i) $2n\pi \pm \dfrac{\pi}{4}$. (ii) $(8n+5)\dfrac{\pi}{4}$ and $(8n+7)\dfrac{\pi}{4}$.

4. $2\sin\frac{1}{2}A = \sqrt{1+\sin A} + \sqrt{1-\sin A}$. 10. $\pm\sqrt{\dfrac{4-(a^2+b^2)}{a^2+b^2}}$.

---

*Known as *Vieta's expression for* π.

CHAPTER III

INVERSE CIRCULAR FUNCTIONS

18. The equation $\sin\theta = x$ means that θ is an angle whose sine is x. It is often convenient to express this statement *inversely* by writing $\theta = \sin^{-1}x$. Thus, the symbol $\sin^{-1}x$ denotes an angle whose sine is x. Hence, $\sin^{-1}x$ is an angle whereas $\sin\theta$ is a number. The two relations $\sin\theta = x$ and $\theta = \sin^{-1}x$ are identical; if one is given, the other follows. The symbol $\sin^{-1}x$ is usually read as "*sine inverse x*" or "*sine minus one x*".

Note. $\sin^{-1}x$ must not be confused with $(\sin x)^{-1}$ *i.e.*, $\dfrac{1}{\sin x}$.

19. We know that if θ be the least positive angle whose sine is equal to x, then all the angles given by $n\pi + (-1)^n\theta$ have their sine equal to x. Hence, $\sin^{-1}x$ has got an infinite number of values, and as such, $\sin^{-1}x$ is a *multiple-valued function*.

Hence, the *general value* of $\sin^{-1}x = n\pi + (-1)^n \sin^{-1}x$ where on the right-hand side, $\sin^{-1}x$ stands for any particular angle whose sine is x.

Similarly, the *general value of*

$$\cos^{-1}x = 2n\pi \pm \cos^{-1}x$$

and of $\tan^{-1}x = n\pi + \tan^{-1}x$.

The smallest numerical value, either positive or negative of θ, where $\sin\theta = x$, is called the *principal value of* $\sin^{-1}x$. Thus, the principal value of $\sin^{-1}\frac{1}{2}$ is $30°$. If corresponding to the same inverse function, there are two angles as found above, one positive and the other negative, it is customary to take the positive angle as the principal value; thus the principal value of $\cos^{-1}\frac{1}{2}$ is $60°$ and not $-60°$, although $\cos(-60°) = \frac{1}{2}$.

In all numerical examples, the principal value is generally taken.

$\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ have similar significance and properties as of $\sin^{-1}x$ mentioned above. These expressions are called **Inverse circular functions.**

20. From the definition, it at once follows that

$$\theta=\sin^{-1}\sin\theta, \text{ and } x=\sin\sin^{-1}x.$$

For if $\sin\theta=x$, then $\theta=\sin^{-1}x=\sin^{-1}\sin\theta$,

and $x=\sin\theta=\sin\sin^{-1}x$.

Similarly, $\theta=\cos^{-1}\cos\theta=\tan^{-1}\tan\theta$; etc.

and $x=\cos\cos^{-1}x=\tan\tan^{-1}x$; etc.

Also, we have

$$\mathbf{cosec^{-1}x=sin^{-1}\frac{1}{x}\,;\; cot^{-1}x=tan^{-1}\frac{1}{x}\,;\; sec^{-1}x=cos^{-1}\frac{1}{x}}.$$

Let $\text{cosec}^{-1}x=\theta$; then $\text{cosec}\,\theta=x$.

$$\therefore\quad \sin\theta=\frac{1}{\text{cosec}\,\theta}=\frac{1}{x}.$$

Hence, $\theta=\sin^{-1}\frac{1}{x}$, and therefore, $\text{cosec}^{-1}x=\sin^{-1}\frac{1}{x}$.

In the same way, we have, $\text{cosec}^{-1}\frac{1}{x}=\sin^{-1}x$.

The other relations follow similarly.

21. As all the trigonometrical ratios can be expressed in terms of any one, similarly all the inverse trigonometrical functions can be expressed in terms of any one of them.

Thus, let $\sin^{-1}x=\theta$; then $\sin\theta=x$.

$$\therefore\quad \cos\theta=\sqrt{1-x^2}\,;\; \tan\theta=\frac{x}{\sqrt{1-x^2}}\,;\; \cot\theta=\frac{\sqrt{1-x^2}}{x}\,;$$

$$\sec\theta=\frac{1}{\sqrt{1-x^2}} \text{ and } \text{cosec}\,\theta=\frac{1}{x}.$$

$$\theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}.$$

22. *To prove that*

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$.

Let $\sin^{-1}x = \theta$; then $\sin\theta = x$. Now, $\sin\theta = \cos(\frac{1}{2}\pi - \theta)$.

$\therefore \quad \cos(\frac{1}{2}\pi - \theta) = x$ and hence $\cos^{-1}x = \frac{1}{2}\pi - \theta$.

Therefore, $\sin^{-1}x + \cos^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$.

Similarly, the other two relations follow.

Note. In the above results, principal values of inverse functions have been taken and x has been tacitly assumed to be positive. If x be negative, results (i) and (iii) will still hold but not (ii). [See *Ex. III, Sum no. 34(ii).*]

23. *To prove that*

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$.

Let $\tan^{-1}x = \alpha$, then $\tan\alpha = x$;

also let $\tan^{-1}y = \beta$, then $\tan\beta = y$.

Now, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{x+y}{1-xy}$.

$$\therefore \quad \alpha + \beta = \tan^{-1}\frac{x+y}{1-xy},$$

$$\text{i.e.,}\quad \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$$

Similarly, the second relation follows.

Note 1. It can be easily proved as above that

$$\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\frac{xy \mp 1}{y \pm x}.$$

Note 2. In the above results we have considered principal values only and have tacitly assumed x, y, $1-xy$ all positive. In case these restrictions are removed, there will be modifications in the results. [*cf. Ex. III, Sum no. 34(i).*]

24. *To prove that*

$$\tan^{-1}\mathbf{x} + \tan^{-1}\mathbf{y} + \tan^{-1}\mathbf{z} = \tan^{-1}\frac{\mathbf{x}+\mathbf{y}+\mathbf{z}-\mathbf{xyz}}{1-\mathbf{yz}-\mathbf{zx}-\mathbf{xy}}.$$

Let $\tan^{-1}x = \alpha$; $\tan^{-1}y = \beta$; $\tan^{-1}z = \gamma$;

$\therefore \quad \tan\alpha = x$, $\tan\beta = y$, $\tan\gamma = z$.

Now, $\tan(\alpha+\beta+\gamma)$

$$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta}$$

$$= \frac{x+y+z-xyz}{1-yz-zx-xy}.$$

Hence, $\alpha + \beta + \gamma = \tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}$.

Since, $\alpha+\beta+\gamma = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z$, the required result follows.

Note. This relation can also be deduced by applying twice the formula of Art. 23. Thus,

left side $= (\tan^{-1}x + \tan^{-1}y) + \tan^{-1}z$

$= \tan^{-1}\dfrac{x+y}{1-xy} + \tan^{-1}z$; now again apply Art. 23.

25. In fact for most of the formulæ involving ordinary circular functions, corresponding relations connecting the inverse circular functions can be easily deduced. In addition to those given above, some are illustrated in the following examples.

Ex. 1. *Show that*

(i) $\sin^{-1}x \pm \cos^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$.

(ii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{(1-x^2)(1-y^2)}\}$.

(i) Let $\sin^{-1}x=\alpha$. $\therefore$ $\sin\alpha=x$ and $\cos\alpha=\sqrt{1-x^2}$;

$\sin^{-1}y=\beta$. $\therefore$ $\sin\beta=y$ and $\cos\beta=\sqrt{1-y^2}$.

Now, $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

$$=x\sqrt{1-y^2}\pm y\sqrt{1-x^2}.$$

$$\therefore\ \alpha\pm\beta=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}.$$

Since, $\alpha\pm\beta=\sin^{-1}x\pm\sin^{-1}y$, the required result follows.

(ii) These relations follow similarly from the values of $\cos(\alpha\pm\beta)$.

Ex. 2. *Show that*

(*i*) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$.

(*ii*) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$.

(*iii*) $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

(i) Let $\sin^{-1}x=\theta$. $\therefore$ $\sin\theta=x$, $\cos\theta=\sqrt{1-x^2}$.

Now, $\sin 2\theta=2\sin\theta\cos\theta=2x\sqrt{1-x^2}$.

$$\therefore\quad 2\theta=\sin^{-1}(2x\sqrt{1-x^2}).$$

Since, $\theta=\sin^{-1}x$ the required result follows.

(ii) & (iii). These relations follow similarly from the corresponding values of $\cos 2\theta$ in terms of $\cos\theta$ and $\tan 2\theta$ in terms of $\tan\theta$.

Note. The above three relations can also be deduced by putting x for y in the values of $\sin^{-1}x+\sin^{-1}y$, $\cos^{-1}x+\cos^{-1}y$ and $\tan^{-1}x+\tan^{-1}y$.

Ex. 3. *Show that*

(*i*) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$.

(*ii*) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$.

(*iii*) $3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$.

(i) Let $\sin^{-1}x=\theta$; then $\sin\theta=x$.

Now, $\sin 3\theta=3\sin\theta-4\sin^3\theta=3x-4x^3$.

$\therefore$ 3θ, *i.e.*, $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$.

(ii) & (iii). These relations follow similarly from the corresponding values of $\cos 3\theta$ in terms of $\cos\theta$ and $\tan 3\theta$ in terms of $\tan\theta$.

Note. The result of (iii) may also be deduced by putting $y=z=x$ in the formula of Art. 24.

Ex. 4. *Show that*

$$2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}=\tan^{-1}\frac{2x}{1-x^2}.$$

Let $\tan^{-1}x=\theta$. $\therefore$ $\tan\theta=x$.

Since, $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{2x}{1+x^2}$.

$\therefore$ 2θ, *i.e.*, $2\tan^{-1}x=\sin^{-1}\dfrac{2x}{1+x^2}$.

Since, $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-x^2}{1+x^2}$,

and $\tan 2\theta=\dfrac{2\tan^2\theta}{1-\tan^2\theta}=\dfrac{2x}{1-x^2}$,

the remaining relations follow similarly.

Ex. 5. *Write down the value of*

$$\tan^{-1}\sin\cos^{-1}\sqrt{\tfrac{2}{3}}.$$

Let $\theta=\tan^{-1}\sin\cos^{-1}\sqrt{\frac{2}{3}}$;

then $\tan\theta=\sin\cos^{-1}\sqrt{\frac{2}{3}}=\sin\sin^{-1}\sqrt{1-\frac{2}{3}}=\sin\sin^{-1}\dfrac{1}{\sqrt{3}}=\dfrac{1}{\sqrt{3}}$.

$\therefore$ the least positive value of $\theta=\frac{1}{6}\pi$

and hence, the general value of θ *i.e.*, of the given expression is

$$n\pi+\tfrac{1}{6}\pi \text{ or } \tfrac{1}{6}\pi(6n+1).$$

Ex. 6. *Show that*

$$2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{32}{43}.$$

Since, $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$, [See *Ex.* 4.]

$$\therefore\ 2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{5^2}}=\tan^{-1}\frac{5}{12}.$$

$$\therefore\ \text{left side}=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{\frac{5}{12}+\frac{1}{4}}{1-\frac{5}{48}}=\tan^{-1}\frac{32}{43}.$$

Ex. 7. *Solve*

$$\sin^{-1}\frac{2a}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}=\tan^{-1}\frac{2x}{1-x^2}.$$

Left side $=2\tan^{-1}a-2\tan^{-1}b$. [See *Ex. 4.*]

$\therefore$ the equation reduces to

$$2\tan^{-1}x=2\tan^{-1}a-2\tan^{-1}b.$$

$$\therefore\quad \tan^{-1}x=\tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}.$$

$$\therefore\quad x=\frac{a-b}{1+ab}.$$

Ex. 8. *Solve*

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}.\qquad [\textit{ C. P. 1947.}]$$

$$\text{Left side}=\tan^{-1}\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x^2-1}{x^2-4}}=\tan^{-1}\frac{2x^2-4}{-3}.$$

$\therefore$ the equation reduces to

$$\tan^{-1}\frac{2x^2-4}{-3}=\frac{\pi}{4},\text{ or, }\frac{2x^2-4}{-3}=\tan\frac{\pi}{4}=1,$$

whence, $x=\pm\dfrac{1}{\sqrt{2}}$.

EXAMPLES III

Prove (*Ex. 1 to Ex. 15*) that :

1. $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

[*C. P. 1935.*]

2. $\tan^{-1}a-\tan^{-1}c=\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}$.

3. (i) $\tan(2\tan^{-1}x)=2\tan(\tan^{-1}x+\tan^{-1}x^3)$.

[*C. P. 1944.*]

(ii) $\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)=\dfrac{2b}{a}$.

[*C. P. 1948.*]

4. $\tan^{-1}x=2\tan^{-1}(\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x)$.

[*C. P. 1938.*]

5. $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$
$$= 2(\tan^{-1} 1 + \tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3}).$$ [*C. H. 1950.*]

6. $\tan^{-1}(\frac{1}{2} \tan 2A) + \tan^{-1} (\cot A) + \tan^{-1} (\cot^3 A) = 0.$

7. $\sin^{-1} \dfrac{4}{5} + \sin^{-1} \dfrac{5}{13} + \sin^{-1} \dfrac{16}{65} = \dfrac{\pi}{2}.$ [*C. H. 1947.*]

8. $\tan^{-1} \dfrac{yz}{xr} + \tan^{-1} \dfrac{zx}{yr} + \tan^{-1} \dfrac{xy}{zr} = \dfrac{\pi}{2},$
where $r^2 = x^2 + y^2 + z^2$.

9. $\tan \left\{\dfrac{1}{2} \sin^{-1} \dfrac{2x}{1+x^2} + \dfrac{1}{2} \cos^{-1} \dfrac{1-x^2}{1+x^2}\right\} = \dfrac{2x}{1-x^2}.$
[*C. P. 1947.*]

10. $\cot^{-1} \dfrac{xy+1}{x-y} + \cot^{-1} \dfrac{yz+1}{y-z} + \cot^{-1} \dfrac{zx+1}{z-x} = 0.$

11. $\tan^{-1} \sqrt{\dfrac{xr}{yz}} + \tan^{-1} \sqrt{\dfrac{yr}{zx}} + \tan^{-1} \sqrt{\dfrac{zr}{xy}} = \pi,$
where $r = x + y + z$. [*C. P. 1935.*]

12. $\dfrac{1}{6} \tan^{-1} \dfrac{2x}{1-x^2} + \dfrac{1}{9} \tan^{-1} \dfrac{3x-x^3}{1-3x^2}$
$$+ \frac{1}{12} \tan^{-1} \frac{4x-4x^3}{1-6x^2+x^4} = \tan^{-1} x.$$

13. $\cos \tan^{-1} \sin \cot^{-1} x = \left(\dfrac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}.$

14. $2 \tan^{-1} \left(\sqrt{\dfrac{a-b}{a+b}} \tan \dfrac{x}{2}\right) = \cos^{-1} \dfrac{b + a \cos x}{a + b \cos x}.$
[*C. P. 1936.*]

15. $\tan^{-1} x + \tan^{-1} y = \dfrac{1}{2} \sin^{-1} \left\{\dfrac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right\}.$

16. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, show that
$x^2 + y^2 + z^2 + 2xyz = 1.$ [*C. H. 1950 ; C. P. 1934, '41.*]

17. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, show that
$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$$
[*C. H. 1937.*]

18. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that

$$x+y+z=xyz.$$ [*C. P. 1940.*]

19. If α, β, γ be the roots of the equation $x^3+px^2+qx+p=0$, prove that $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=n\pi$ radians, except in one particular case. [*C. P. 1940.*]

20. Solve the following equations :

(i) $\sin^{-1}\dfrac{2a}{1+a^2}+\sin^{-1}\dfrac{2b}{1+b^2}=2\tan^{-1}x$. [*C. P. 1936.*]

(ii) $\tan^{-1}2x+\tan^{-1}3x=\frac{1}{4}\pi$.

(iii) $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$. [*C. P. 1933.*]

(iv) $\sin 2\cos^{-1}\cot 2\tan^{-1}x=0$.

(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2}=2\sec^{-1}\sqrt{1+a^2}-2\sec^{-1}\sqrt{1+b^2}$.

(vi) $\cos^{-1}(x+\frac{1}{2})+\cos^{-1}x+\cos^{-1}(x-\frac{1}{2})=\frac{3}{2}\pi$. [*C. H. 1931.*]

(vii) $\sin^{-1}x+\sin^{-1}y=\frac{2}{3}\pi$;
$\cos^{-1}x-\cos^{-1}y=\frac{1}{3}\pi$. [*C. P. 1940.*]

(viii) $\tan^{-1}\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=a$.

(ix) $\cot^{-1}(x-1)+\cot^{-1}(x-2)+\cot^{-1}(x-3)=0$. [*C. H. 1939.*]

(x) $\tan^{-1}\dfrac{a}{x}+\tan^{-1}\dfrac{b}{x}+\tan^{-1}\dfrac{c}{x}+\tan^{-1}\dfrac{d}{x}=\dfrac{\pi}{2}$.

(xi) $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}x=\frac{1}{4}\pi$. [*C. P. 1931.*]

21. Find all the positive integral solutions of

$$\tan^{-1}x+\cot^{-1}y=\tan^{-1}3.$$

22. If k be a positive integer, show that the equation

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}k$$

has no positive integral solutions.

23. (i) If $\sin(\pi\cos a) = \cos(\pi\sin a)$, show that

$$a = \pm\tfrac{1}{2}\sin^{-1}\tfrac{3}{4}.$$

(ii) If $\tan(\pi\cos\tfrac{1}{2}\theta) = \cot(\pi\sin\tfrac{1}{2}\theta)$, express $\sin\theta$ in its simplest form. [*C. P. 1941.*]

24. If $\tan(\pi\cot a) = \cot(\pi\tan a)$, show that

$$\tan a = \tfrac{1}{4}\{(2n+1) \pm \sqrt{4n^2+4n-15}\},$$

where n is any integer except $1, 0, -1, -2$.

25. Write down the general values of

$$\sin^{-1}\tfrac{1}{2}(-1)^k,\ \cos^{-1}\tfrac{1}{2}(-1)^k,\ \tan^{-1}(-1)^k,$$

where k is an integer.

26. Show that

$$\sin^{-1}\frac{2a_1b_1}{a_1^2+b_1^2} + \sin^{-1}\frac{2a_2b_2}{a_2^2+b_2^2} + \cdots + \sin^{-1}\frac{2a_nb_n}{a_n^2+b_n^2}$$

can be expressed in the form $\sin^{-1}\dfrac{2xy}{x^2+y^2}$, where x and y are rational functions of $a_1, b_1, a_2, b_2, \ldots a_n, b_n$.

27. Show that

$$\tan^{-1}\frac{a_1x-y}{a_1y+x} + \tan^{-1}\frac{a_2-a_1}{a_1a_2+1} + \tan^{-1}\frac{a_3-a_2}{a_2a_3+1} + \cdots$$
$$+ \tan^{-1}\frac{a_n-a_{n-1}}{a_{n-1}a_n+1} + \tan^{-1}\frac{1}{a_n} = \tan^{-1}\frac{x}{y},$$

where $a_1, a_2, \ldots\ldots\ldots a_n$ are any quantities.

28. Given that $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are in A. P. point out the existence of a certain *algebraic* relation between x, y, z. If in addition x, y, z are also in A. P., prove that $x = y = z$. [*C. H. 1930.*]

29. Solve the equation

$$\theta = \tan^{-1}(2\tan^2\theta) - \tfrac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right).$$

[*C. H. 1933.*]

30. Prove that $\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}+\tan^{-1}\dfrac{c-a}{1+ca}$

$$=\tan^{-1}\frac{a^2-b^2}{1+a^2b^2}+\tan^{-1}\frac{b^2-c^2}{1+b^2c^2}+\tan^{-1}\frac{c^2-a^2}{1+c^2a^2}.$$

[*C. P. 1931.*]

31. Prove that

$$\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z)$$
$$=\cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z).$$ [*C. P. 1939.*]

32. Find the general value of $\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$.

33. Solve for x and y

$$\tan^{-1}x-\tan^{-1}y=\cot^{-1}2y-\cot^{-1}2x=\tfrac{1}{6}\pi.$$

34. (i) Show that the value of n in the formula

$$\tan^{-1}x+\tan^{-1}y=n\pi+\tan^{-1}\frac{x+y}{1-xy}$$

is zero, unless $xy>1$ and if $xy>1$, n is $+1$ or -1 according as x and y are both positive or both negative (principal values only being considered).

(ii) Show that $\tan^{-1}x+\cot^{-1}x=-\dfrac{\pi}{2}$, if $x<0$.

35. Find if there is any value of x which strictly satisfies the equation

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7).$$

[*C. H. 1943.*]

36. If $\tan^{-1}\sqrt{\left(\dfrac{a-c}{c+x}\right)}+\tan^{-1}\sqrt{\left(\dfrac{a-c}{c+y}\right)}$

$$+\tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)}=0,$$

show that

$$\begin{vmatrix} 1 & x & (a+x)\sqrt{(c+x)} \\ 1 & y & (a+y)\sqrt{(c+y)} \\ 1 & z & (a+z)\sqrt{(c+z)} \end{vmatrix}=0.$$

[*C. H. 1943.*]

ANSWERS

20. (i) $\dfrac{a+b}{1-ab}$. (ii) -1 or $\frac{1}{3}$. (iii) $0, \pm\frac{1}{2}$.

(iv) $\pm 1, \pm 1 \pm \sqrt{2}$. (v) $\dfrac{a-b}{1+ab}$. (vi) 0.

(vii) $x=\frac{1}{2}, y=1$. (viii) $\pm\sqrt{\sin 2a}$. (ix) $\frac{1}{3}(6 \pm \sqrt{6})$.

(x) x is given by the equation

$$x^4 - x^2(ab+ac+ad+bc+bd+cd)+abcd=0.$$

(xi) $\frac{1}{3}$.

21. When $x=1$, $y=2$ and when $x=2$, $y=7$. **23.** (ii) $\sin\theta = -\frac{3}{4}$.

25. $n\pi+(-1)^{k+n}\dfrac{\pi}{6}$, $(2n+k)\pi \pm \dfrac{\pi}{3}$, $n\pi+(-1)^k\dfrac{\pi}{4}$.

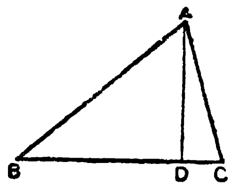
28. $(x+z)(1-y^2)=2y(1-xz)$. **29.** $\tan\theta = 0, \pm 1, -2$. **32.** $k\pi - 2x$.

33. $x=\frac{1}{4}(3\pm\sqrt{17})$, $y=\frac{1}{4}(-3\pm\sqrt{17})$. **35.** No value.

CHAPTER IV

PROPERTIES OF TRIANGLES

26. Area of a triangle.



Let ABC be a triangle and let Δ denote its area. Draw AD perpendicular to BC; then from ΔACD,

$$AD = AC \sin C = b \sin C.$$

Now, $\Delta = \frac{1}{2} BC.AD = \frac{1}{2} ab \sin C.$

Similarly, by drawing perpendiculars from B and C to the opposite sides, it can be shown that

$$\Delta = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \sin B.$$

Otherwise.
$$\left.\begin{aligned} \Delta &= \tfrac{1}{2}\,\mathbf{ab \sin C} \\ &= \tfrac{1}{2}\,\mathbf{ac \sin B}\ (\because\ b \sin C = c \sin B) \\ &= \tfrac{1}{2}\,\mathbf{bc \sin A}\ (\because\ a \sin B = b \sin A) \end{aligned}\right\} \qquad (1)$$

Thus, $\Delta =$ (*half the product of any two sides*) $\times$ *sine of the included angle.*

Again, $\Delta = \frac{1}{2} bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} . \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\mathbf{s(s-a)(s-b)(s-c)}}. \qquad \cdots \qquad \cdots \qquad (2)$$

Substituting in the above expression $s = \frac{1}{2}(a+b+c)$, we get

$$\Delta = \tfrac{1}{4} \sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$

$$= \tfrac{1}{4}\{\mathbf{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}\}^{\frac{1}{2}} \quad (3)$$

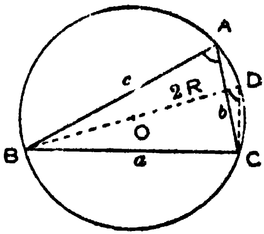
Note 1. In the figure given above, the perpendicular AD falls inside the triangle. If either B or C be obtuse, this would not be

the case. In such a case, if say C be obtuse, $AD = AC \sin (180° - C)$ $= b \sin C$, and the same result follows.

Note 2. In some text books, S is used to denote the area of a triangle ; but to avoid confusion between S and s in writing, the symbol Δ is preferable.

27. Circum-radius of a triangle.

Let O be the centre, and R, the radius of the circle circumscribing the triangle ABC.



Join BO and produce it to meet the circumference in D.

Join CD.

From ΔBCD, $\sin BDC = \dfrac{BC}{BD} = \dfrac{a}{2R}$.

But $\angle BDC = \angle A$, being in the same segment.

$$\therefore \quad \frac{a}{2R} = \sin A, \quad \text{or,} \quad R = \frac{a}{2 \sin A}.$$

Similarly, by joining AO and producing it to meet the circumference in E and joining CE, BE, it can be shown that

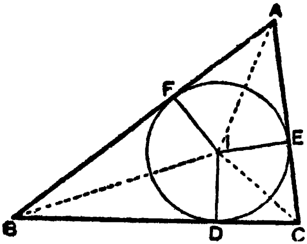
$$R = \frac{b}{2 \sin B} \quad \text{and} \quad \text{also} = \frac{c}{2 \sin C}.$$

Thus, $$\frac{\mathbf{a}}{\mathbf{\sin A}} = \frac{\mathbf{b}}{\mathbf{\sin B}} = \frac{\mathbf{c}}{\mathbf{\sin C}} = \mathbf{2R}. \qquad (4)$$

Again, $$\mathbf{R} = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{\mathbf{abc}}{\mathbf{4\Delta}}. \qquad (5)$$

Note. If angle A be obtuse, A and D fall on the opposite sides of BC and $ABCD$ being cyclic, $\sin BDC = \sin (180° - A) = \sin A$ and the same result follows. In case A is a right angle, evidently $2R = a = \dfrac{a}{\sin A}$ and we get the same result.

28. In-radius of a triangle.



Let I be the centre and r the radius of the circle inscribed in the triangle ABC; let D, E, F be the points of contact of the in-circle with the sides BC, CA, AB respectively.

Then, $ID = IE = IF = r$.

Join IA, IB, IC.

$$\begin{aligned}\Delta ABC &= \Delta IBC + \Delta ICA + \Delta IAB\\ &= \tfrac{1}{2}BC.ID + \tfrac{1}{2}CA.IE + \tfrac{1}{2}AB.IF\\ &= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr\\ &= \tfrac{1}{2}r(a+b+c) = rs.\end{aligned}$$

Thus, $\Delta = rs$.

$$\mathbf{r = \frac{\Delta}{s}}. \qquad (6)$$

Again, $a = BC = BD + DC$

$= r \cot \tfrac{1}{2}B + r \cot \tfrac{1}{2}C$, from Δ^s IBD, ICD,

$$= r\left[\frac{\cos\frac{1}{2}B}{\sin\frac{1}{2}B} + \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}\right]$$

$$= r\left[\frac{\cos\frac{1}{2}B\sin\frac{1}{2}C + \sin\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}B\sin\frac{1}{2}C}\right]$$

$$= r\frac{\sin(\frac{1}{2}B+\frac{1}{2}C)}{\sin\frac{1}{2}B\sin\frac{1}{2}C} = r\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}$$

[$\because$ $\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90°$,

$\therefore$ $\sin(\tfrac{1}{2}B + \tfrac{1}{2}C) = \sin(90° - \tfrac{1}{2}A) = \cos\tfrac{1}{2}A$]

$\therefore$ $r = a \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C \sec\tfrac{1}{2}A$.

Since by (4), $a = 2R\sin A = 4R\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$,

$$\therefore \quad \mathbf{r = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.} \qquad \ldots (7)$$

Since, from the fig., $AF = AE$, $BD = BF$, $CD = CE$ and since the sum of these six quantities is equal to the perimeter,

$\therefore \quad AF + BD + CD =$ semi-perimeter $= s$,

i.e., $\quad AF + BC$, or, $AF + a = s$,

$\therefore \quad AF = s - a = AE$.

Similarly, $BF = s - b = BD$; $CE = s - c = CD$.

From $\triangle AIF$, $IF = AF \tan IAF$.

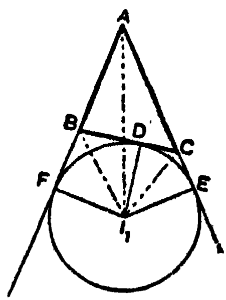
$$\left.\begin{array}{rl} \therefore & \mathbf{r = (s-a) \tan \tfrac{1}{2}A.} \\ \text{Similarly,} & \mathbf{r = (s-b) \tan \tfrac{1}{2}B,} \\ \text{and} & \mathbf{r = (s-c) \tan \tfrac{1}{2}C.} \end{array}\right\} \quad \cdots \quad (8)$$

Note. ***Distances of the in-centres from the vertices.***

From $\triangle AIF$, $IA = IF$ cosec IAF, $\therefore$ ***$IA = r$ cosec $\frac{1}{2}A$.***

Similarly, ***$IB = r$ cosec $\frac{1}{2}B$*** and ***$IC = r$ cosec $\frac{1}{2}C$.***

29. Ex-radii of a triangle.



Let I_1 be the centre, and r_1 the radius of the escribed circle (opposite to the angle A) of the $\triangle ABC$; let D, E, F be the points of contact of the circle with the sides BC, and AC and AB produced.

Let r_2, r_3 denote the radii of the escribed circles opposite to the angles B and C respecttively.

Now, $I_1D = I_1E = I_1F = r_1$; join AI_1, BI_1, CI_1.

$$\begin{aligned} \triangle ABC &= \triangle I_1AB + \triangle I_1AC - \triangle I_1BC \\ &= \tfrac{1}{2}AB.\, I_1F + \tfrac{1}{2}AC.\, I_1E - \tfrac{1}{2}BC.\, I_1D \\ &= \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1 \\ &= \tfrac{1}{2}r_1 (b + c - a) \\ &= \tfrac{1}{2}r_1 (b + c + a - 2a) \\ &= \tfrac{1}{2}r_1 (2s - 2a) \\ &= r_1 (s - a). \end{aligned}$$

Thus, $\quad \Delta = r_1 (s-a)$.

$$\left.\begin{aligned} \therefore \quad \mathbf{r_1} &= \frac{\Delta}{\mathbf{s-a}} \\ \text{Similarly,} \quad \mathbf{r_2} &= \frac{\Delta}{\mathbf{s-b}} \\ \mathbf{r_3} &= \frac{\Delta}{\mathbf{s-c}} \end{aligned}\right\} \tag{9}$$

Again, $a = BC = BD + CD$

$$= r_1 \cot I_1BD + r_1 \cot I_1CD,$$

from $\Delta^s\ I_1BD,\ I_1CD$

$$= r_1 \cot (90^\circ - \tfrac{1}{2}B) + r_1 \cot (90^\circ - \tfrac{1}{2}C)$$

because, $\angle I_1BD = \frac{1}{2}(180^\circ - B) = 90^\circ - \frac{1}{2}B$

and $\quad \angle I_1CD = \frac{1}{2}(180^\circ - C) = 90^\circ - \frac{1}{2}C$.

$$\therefore \quad a = r_1 (\tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C)$$

$$= r_1 \left[\frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B} + \frac{\sin \frac{1}{2}C}{\cos \frac{1}{2}C}\right]$$

$$= r_1 \left[\frac{\sin \frac{1}{2}B \cos \frac{1}{2}C + \sin \frac{1}{2}C \cos \frac{1}{2}B}{\cos \frac{1}{2}B \cos \frac{1}{2}C}\right]$$

$$= r_1 \frac{\sin (\frac{1}{2}B + \frac{1}{2}C)}{\cos \frac{1}{2}B \cos \frac{1}{2}C}$$

$$= r_1 \frac{\cos \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C}, \text{ as in Art. 28.}$$

$$\therefore \quad r_1 = a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \sec \tfrac{1}{2}A.$$

Putting $\quad a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$,

$$\left.\begin{aligned} \mathbf{r_1} &= \mathbf{4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.} \\ \text{Similarly,} \quad \mathbf{r_2} &= \mathbf{4R \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C,} \\ \text{and} \quad \mathbf{r_3} &= \mathbf{4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C.} \end{aligned}\right\} \tag{10}$$

Again, $\quad AE = AC + CE = b + CD \qquad (\because \quad CE = CD)$

and $\quad AF = AB + BF = c + BD. \qquad (\because \quad BF = BD)$

But $AE = AF$; therefore, by addition, we get

$$2AE = b + c + BD + CD = b + c + a = 2s.$$

$$\therefore \quad AE = s.$$

Again, from $\triangle AI_1E$, $I_1E = AE \tan I_1AE$.

$$\left.\begin{array}{ll} \therefore & \mathbf{r_1 = s \tan \tfrac{1}{2}A.} \\ \text{Similarly,} & \mathbf{r_2 = s \tan \tfrac{1}{2}B,} \\ \text{and} & \mathbf{r_3 = s \tan \tfrac{1}{2}C.} \end{array}\right\} \quad \cdots \quad (11)$$

Note. ***Distances of ex-centres from the vertices.***

From $\triangle AI_1F$, $I_1A = I_1F \operatorname{cosec} I_1AF$.

$$\therefore \quad \boldsymbol{I_1A = r_1 \operatorname{cosec} \tfrac{1}{2}A}$$

$$\boldsymbol{= 4R \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}, \text{ by formula (10).}$$

From $\triangle BI_1F$, $I_1B = I_1F \operatorname{cosec} I_1BF$.

$$\therefore \quad \boldsymbol{I_1B = r_1 \sec \tfrac{1}{2}B.} \quad (\because \angle I_1BF = 90^\circ - \tfrac{1}{2}B)$$

Similarly, $\boldsymbol{I_1C = r_1 \sec \tfrac{1}{2}C.}$

In the same way, $\boldsymbol{I_2B = r_2 \operatorname{cosec} \tfrac{1}{2}B}$, $\boldsymbol{I_3C = r_3 \operatorname{cosec} \tfrac{1}{2}C}$.

Ex. 1. *Prove that* $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$. [*C. P. 1938, '43, '50.*]

By formula (9),

$$\text{left side} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{3s-(a+b+c)}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

Ex. 2. *Prove that* $4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C = \dfrac{s}{R}$.

[*C. P. 1930, '44*]

$$\text{Left side} = 4\sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ca}} \cdot \sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{4s}{abc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{4s}{abc}\Delta = s \cdot \frac{4\Delta}{abc} = \frac{s}{R}, \text{ by formula (5).}$$

Ex. 3. *Show that*

$$\frac{bc-r_2r_3}{r_1}=\frac{ca-r_3r_1}{r_2}=\frac{ab-r_1r_2}{r_3}.$$

$$r_2r_3=\frac{\Delta^2}{(s-b)(s-c)}=s(s-a).$$

$$\begin{aligned}\therefore\quad bc-r_2r_3&=\tfrac{1}{4}[4bc-2s(2s-2a)]\\&=\tfrac{1}{4}[4bc-(a+b+c)(b+c-a)]\\&=\tfrac{1}{4}[4bc+a^2-(b+c)^2]\\&=\tfrac{1}{4}[a^2-(b-c)^2]\\&=\tfrac{1}{4}[(a+b-c)(a-b+c)]\\&=(s-b)(s-c).\end{aligned}$$

$$\therefore\quad \frac{bc-r_2r_3}{r_1}=\frac{(s-b)(s-c)}{r_1}=\frac{(s-a)(s-b)(s-c)}{\Delta}=\frac{\Delta}{s}=r.$$

Similarly, the other ratios are equal to the same quantity.

Ex. 4. *If α, β, γ be the distances of the angular points of a triangle from the points of contact of the in-circle with the sides, show that*

$$r=\left(\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}.$$ [*C. H.* 1924, '25, '29, '51.]

From the fig. of Art. 28, we have,

$$AF=AE=\alpha\ ;\ BF=BD=\beta\ ;\ CD=CE=\gamma.$$

$$\therefore\quad \alpha=s-a,\ \beta=s-b,\ \gamma=s-c\ ;$$

hence, $\alpha+\beta+\gamma=3s-(a+b+c)=3s-2s=s.$

From formula (6), $$r^2=\frac{\Delta^2}{s^2}=\frac{s(s-a)(s-b)(s-c)}{s^2}$$

$$=\frac{(s-a)(s-b)(s-c)}{s}$$

$$=\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}.$$

Hence, the required result follows.

Ex. 5. *If $r:r_1:R=2:12:5$, show that the triangle is right angled.*

From the given relation, we have

$$\begin{aligned}2R=r_1-r&=4R\sin\tfrac{1}{2}A\,(\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C-\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C)\\&=4R\sin\tfrac{1}{2}A\cos(\tfrac{1}{2}B+\tfrac{1}{2}C)\\&=4R\sin^2\tfrac{1}{2}A=2R\,(1-\cos A).\end{aligned}$$

$\cos A=0$, or, $A=90^\circ$.

EXAMPLES IV(a)

1. Find the area of the triangle whose sides are

$$\frac{y}{z}+\frac{z}{x},\ \frac{z}{x}+\frac{x}{y},\ \frac{x}{y}+\frac{y}{z}.$$

2. Prove that

(i) $b^2 \sin 2C + c^2 \sin 2B = 4\Delta$.

(ii) $r_1 + r_2 + r_3 - r = 4R$.

(iii) $r_2r_3 + r_3r_1 + r_1r_2 = s^2$. [*C. P. 1943.*]

(iv) $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$.

(v) $\sin A + \sin B + \sin C = \dfrac{s}{R}$. [*C. P. 1951.*]

(vi) $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$.

(vii) $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.

(viii) $R = \dfrac{1}{4}\,\dfrac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}$.

(ix) $\Delta = \sqrt{rr_1r_2r_3} = r^2 \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$.

(x) $a \cot A + b \cot B + c \cot C = 2(R + r)$.
[*C. P. 1949.*]

(xi) $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$.
[*C. P. 1934.*]

(xii) $a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \dfrac{\Delta}{R}$.

(xiii) $r_1 (\cos B - \cos C) + r_2 (\cos C - \cos A)$
$+ r_3 (\cos A - \cos B) = 0$.

(xiv) $r_1 = R(\cos B + \cos C - \cos A + 1)$.

(xv) $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{4R}{r^2 s^2}$.

(xvi) $\left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^2 = \frac{4}{r}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$.

(xvii) $r_1(r_2 + r_3)\operatorname{cosec} A = r_2(r_3 + r_1)\operatorname{cosec} B$
$= r_3(r_1 + r_2)\operatorname{cosec} C$.

(xviii) $\sin A = 2r_1 \frac{\sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}}{(r_1 + r_2)(r_1 + r_3)}$.

(xix) $\cos A = \frac{2R + r - r_1}{2R}$. [*C. P. 1945, '47.*]

(xx) $\frac{bc}{r_1} + \frac{ca}{r_2} + \frac{ab}{r_3} = 2R\left\{\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3\right\}$.

3. If a, b, c are in A. P., show that r_1, r_2, r_3 are in H. P.

4. (i) If $r_1 = r_2 + r_3 + r$, prove that the triangle is right-angled. [*C. P. 1946.*]

(ii) If $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right-angled.

5. Prove that the circum-radius of $\triangle OBC \geqslant \frac{1}{2}R$.

6. If S be the area of the in-circle, S_1, S_2, S_3, the areas of the escribed circles, then

$$\frac{1}{\sqrt{S}} = \frac{1}{\sqrt{S_1}} + \frac{1}{\sqrt{S_2}} + \frac{1}{\sqrt{S_3}}.$$ [*C. P. 1948.*]

7. Prove that r_1, r_2, r_3 are the roots of the cubic

$$x^3 - x^2(4R + r) + s^2 x - rs^2 = 0.$$

[*Use Ex. 2(ii) and (iii).*]

8. If I be the in-centre of the triangle ABC and x, y, z be the circum-radii of the triangles IBC, ICA, IAB, show that $4R^3 - R(x^2 + y^2 + z^2) - xyz = 0$.

[$x = 2R \sin \frac{1}{2}A.$]

9. Show that in any triangle,

$$a = K^{-1} r_1 (r_2 + r_3)$$
$$b = K^{-1} r_2 (r_3 + r_1)$$
$$c = K^{-1} r_3 (r_1 + r_2)$$

where $K = (r_2 r_3 + r_3 r_1 + r_1 r_2)^{\frac{1}{2}}$. [*C. H. 1927.*]

10. If the in-circle touches the sides of the triangle ABC in L, M, N and if x, y, z be the circum-radii of the triangles MIN, NIL, LIM, prove that $2xyz = Rr^2$.

$$\left[2x = \frac{MN}{\sin MIN} = \frac{2r \cos \frac{1}{2}A}{\sin (180^\circ - A)} = \frac{r}{\sin \frac{1}{2}A}\right]$$

11. In any triangle,

(i) (*a*) if $\cos A = \dfrac{\sin B}{2 \sin C}$, the triangle is isosceles;

(*b*) if OI be parallel to BC, $\cos B + \cos C = 1$.

(ii) prove that

$$a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$$

[*C. H. 1942.*]

12. Given that the sides of a triangle are in A. P. and the greatest angle exceeds the least by 90°, show that the sides are $\sqrt{7} + 1 : \sqrt{7} : \sqrt{7} - 1$. [*C. H. 1931.*]

13. If X, Y, Z be the middle points of the arcs BC, CA, AB of the circum-circle of the triangle ABC, show that the radius of the circle inscribed in the triangle XYZ is

$$4R \sin \frac{B+C}{4} \sin \frac{C+A}{4} \sin \frac{A+B}{4}.$$ [*C. H. 1932.*]

14. If x, y, z are respectively equal to IA, IB, IC and α, β, γ are respectively equal to I_1A, I_2B, I_3C, show that

(i) $\dfrac{xyz}{abc} = \dfrac{r}{s}$. (iii) $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$.

(ii) $\dfrac{bc}{\alpha^2} + \dfrac{ca}{\beta^2} + \dfrac{ab}{\gamma^2} = 1$. (iv) $ax^2 + by^2 + cz^2 = abc$.

15. In any triangle the area of the inscribed circle is to the area of the triangle as $\pi : \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$.

[*C. P. 1931.*]

16. If x, y, z be the lengths of the perpendiculars from the circum-centre on the sides a, b, c, prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

17. If l, m, n are the perpendiculars from the angular points of a triangle upon the opposite sides a, b, c, show that

$$\frac{bl}{c} + \frac{cm}{a} + \frac{an}{b} = \frac{\qquad}{2R}$$

18. If in a triangle, $3R = 4r$, show that

$$4(\cos A + \cos B + \cos C) = 7.$$

19. If the diameter of an ex-circle be equal to the perimeter of the triangle, show that the triangle is right-angled.

[*C. P. 1948.*]

20. If $\left(1 - \dfrac{r_1}{r_2}\right)\left(1 - \dfrac{r_1}{r_3}\right) = 2$,

then the triangle is right-angled. [*C. P. 1949.*]

21. If in a triangle, $\cos A + 2 \cos C : \cos A + 2 \cos B = \sin B : \sin C$, prove that the triangle is either isosceles or right-angled.

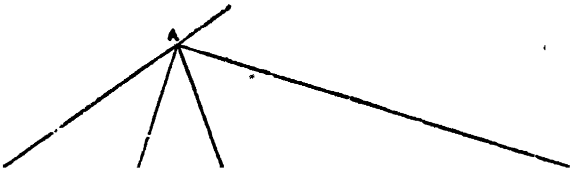
22. If p_1, p_2, p_3 are the perpendiculars from the angular points to the opposite sides, show that

$$p_1^{-1} + p_2^{-1} + p_3^{-1} = r_1^{-1} + r_2^{-1} + r_3^{-1}.$$

ANSWERS

$$\sqrt{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}.$$

30. The bisectors of the angles.



Let D and D' be the points in which the internal and external bisectors of the angle A meet the opposite side BC. Let x denote the length of the internal bisector AD and x' the length of the external bisector AD'.

Since $\triangle ABD + \triangle ACD = \triangle ABC$,

$$\therefore \quad \tfrac{1}{2}xc \sin \tfrac{1}{2}A + \tfrac{1}{2}xb \sin \tfrac{1}{2}A = \tfrac{1}{2}bc \sin A.$$

$$\mathbf{x} = \frac{bc}{b+c} \cdot \frac{\sin A}{\sin \tfrac{1}{2}A}$$

$$= \frac{bc}{b+c} \cdot \frac{2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A}{\sin \tfrac{1}{2}A}$$

$$= \mathbf{\frac{2bc}{b+c} \cos \tfrac{1}{2}A.}$$

Again, since $\triangle ABD' - \triangle ACD' = \triangle ABC$,

$$\therefore \quad \tfrac{1}{2}x'c \sin BAD' - \tfrac{1}{2}x'b \sin CAD' = \triangle ABC,$$

i.e., $\tfrac{1}{2}x'c \cos \tfrac{1}{2}A^* - \tfrac{1}{2}x'b \cos \tfrac{1}{2}A = \tfrac{1}{2}bc \sin A$,

$$\therefore \quad \mathbf{x}' = \frac{bc}{c-b} \cdot \frac{\sin A}{\cos \tfrac{1}{2}A} = \frac{bc}{c-b} \cdot \frac{2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A}{\cos \tfrac{1}{2}A}$$

$$= \mathbf{\frac{2bc}{c-b} \sin \tfrac{1}{2}A.}$$

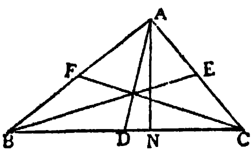
*$\angle DAD' = 90°$. Hence, $\sin BAD' = \sin (90° + \tfrac{1}{2}A) = \cos \tfrac{1}{2}A$ and $\sin CAD' = \sin (90° - \tfrac{1}{2}A) = \cos \tfrac{1}{2}A$.

If y and z denote the lengths of the internal bisectors of the angles B and C, and y' and z', the lengths of the corresponding external bisectors, it can be shown similarly that

$$\mathbf{y}=\frac{2ca}{c+a}\cos\tfrac{1}{2}B\ ;\ \mathbf{z}=\frac{2ab}{a+b}\cos\tfrac{1}{2}C$$

and $$\mathbf{y}'=\frac{2ca}{c\sim a}\sin\tfrac{1}{2}B\ ;\ \mathbf{z}'=\frac{2ab}{a\sim b}\sin\tfrac{1}{2}C.$$

31. The Medians.



Let l, m, n denote the lengths of the medians AD, BE, CF. From Geometry, we have

$$AB^2+AC^2=2(AD^2+BD^2).$$

$$\therefore\quad c^2+b^2=2(l^2+\tfrac{1}{4}a^2),$$

or, $$l^2=\tfrac{1}{2}(b^2+c^2)-\tfrac{1}{4}a^2$$
$$=\tfrac{1}{4}(2b^2+2c^2-a^2)$$
$$=\tfrac{1}{4}(\mathbf{b^2+c^2+2bc\cos A}),$$

by putting $a^2=b^2+c^2-2bc\cos A$.

Similarly, $$\mathbf{m}^2=\tfrac{1}{2}(c^2+a^2)-\tfrac{1}{4}b^2$$
$$=\tfrac{1}{4}(\mathbf{c^2+a^2+2ca\cos B}),$$

and $$\mathbf{n}^2=\tfrac{1}{2}(a^2+b^2)-\tfrac{1}{4}c^2$$
$$=\tfrac{1}{4}(\mathbf{a^2+b^2+2ab\cos C}).$$

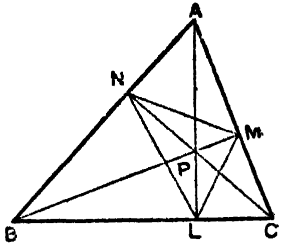
Draw AN perpendicular to BC and let $\angle ADC=\theta$.

Then, $$\cot\theta=\frac{DN}{AN}=\frac{1}{2}\cdot\frac{BN-CN}{AN}$$
$$=\tfrac{1}{2}(\mathbf{\cot B-\cot C}).$$

Again, if $\angle BAD=\beta$ and $\angle CAD=\gamma$, then by drawing perpendiculars from B and C on AD, it can be shown that

$$\cot\theta=\tfrac{1}{2}(\cot\beta-\cot\gamma).$$

32. The pedal triangle.



The triangle LMN formed by joining the feet of the perpendiculars AL, BM, CN, drawn from A, B, C on the opposite sides is called the *pedal triangle* of the $\triangle ABC$. Let P be the orthocentre of $\triangle ABC$.

Since, $\angle^s$ PLB, PNB are right angles, the points* P, N, B, L are concyclic.

$\therefore \quad \angle PLN = \angle PBN$, being in the same segment

$$= 90° - A.$$

Similarly, the points M, P, L, C are concyclic.

$$\therefore \quad \angle PLM = \angle PCM = 90° - A.$$

$$\therefore \quad \angle MLN = 180° - 2A.$$

Thus, the ***angles of the pedal triangle*** are

$$\mathbf{180° - 2A,\ 180° - 2B,\ 180° - 2C.}$$

Again, since $\triangle^s$ AMN, ABC are similar,*

$$\therefore \quad \frac{MN}{BC} = \frac{AN}{AC} = \cos A,$$

or, $\quad MN = a \cos A$.

Thus, the ***sides of the pedal triangle*** are

$$\mathbf{a \cos A,\ b \cos B,\ c \cos C,}$$

or, $R \sin 2A$, $R \sin 2B$, $R \sin 2C$ ($\because$ $a = 2R \sin A$, etc.)

* B, N, M, C are concyclic. $\quad \therefore \quad \angle AMN = 180° - \angle NMC = \angle B$; $\angle A$ is common; hence the similarity.

The circum-radius of the pedal triangle

$$=\frac{MN}{2\sin MLN}$$

$$=\frac{R\sin 2A}{2\sin(180^\circ-2A)}=\tfrac{1}{2}\mathbf{R}.$$

Distances of the ortho-centre from the vertices.

From $\triangle ACN$, $AN=b\cos A$.

From $\triangle\ APN$, $AN=AP\cos PAN$.

From $\triangle LAB$, $\cos PAN=\cos(90^\circ-B)=\sin B$

$\therefore\quad b\cos A=AP\sin B,$

or, $\quad\mathbf{AP}=\dfrac{b\cos A}{\sin B}=\mathbf{2R\cos A}.$

Similarly, $\quad\mathbf{BP=2R\cos B}$ and $\mathbf{CP=2R\cos C}$.

Distances of the ortho-centre from the sides.

$\angle BPL=180^\circ-\angle LPM=\angle C.\ \because\ MPLC$ is cyclic ;

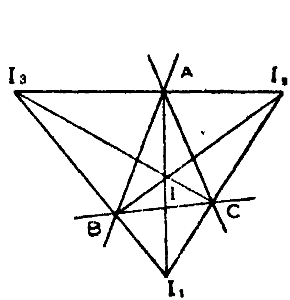
$\therefore\quad \mathbf{PL}=PB\cos BPL=\mathbf{2R\cos B\cos C}.$

Similarly, $\mathbf{PM=2R\cos C\cos A}$; $\mathbf{PN=2R\cos A\cos B}$.

Note. In the course of the above proof, it is assumed that the triangle is acute-angled. If the *angle* A is *obtuse*, the ortho-centre lies outside the triangle and it can be easily shown that the angles of the pedal triangle are $2A-180^\circ$, $2B$, $2C$ and the sides are $-a\cos A$, $b\cos B$, $c\cos C$; the distances of the ortho-centre from the vertices are $-2R\cos A$, $2R\cos B$, $2R\cos C$.

Cor. Since $\angle PLM=\angle PLN$, PL bisects the angle L. Similarly, PM and PN bisects the angles M and N. Therefore P is the in-centre of $\triangle LMN$. Thus, ***the ortho-centre of a triangle is the in-centre of the pedal triangle.*** Since BC is perpendicular to AL, it is the external bisector of $\angle MLN$; hence A, B, C ***are the ex-centres of the pedal triangle.***

33. The Ex-central triangle.



The triangle $I_1I_2I_3$ formed by joining the three ex-centres I_1, I_2, I_3 of $\triangle ABC$ is called the *Ex-central triangle* of $\triangle ABC$.

Let I be the in-centre ; then from the construction for finding the positions of the in-centre and the ex-centres, it follows that

(i) the line AII_1, BII_2, CII_3 bisect the angles A, B, C internally and the straight lines I_2AI_3, I_3BI_1, I_1CI_2 bisect the angles A, B, C externally.

(ii) Hence, AI_1, BI_2, CI_3 are respectively perpendiculars to I_2I_3, I_3I_1, I_1I_2.

(iii) Therefore, *I is the ortho-centre and ABC is the pedal triangle of $\triangle I_1I_2I_3$.* Since ABC is the pedal triangle of $\triangle I_1I_2I_3$, we have from the last article,

$$A = 180° - 2I_1$$

$$\therefore \quad I_1 = 90° - \tfrac{1}{2}A.$$

Thus, the ***angles of the ex-central triangle*** are

$$\mathbf{90° - \tfrac{1}{2}A,\ 90° - \tfrac{1}{2}B,\ 90° - \tfrac{1}{2}C.}$$

Also, we have, BC or $a = I_2I_3 \cos I_1$ [*By Art. 32.*]

$$= I_2I_3 \cos (90° - \tfrac{1}{2}A) ;$$

$$\therefore \quad a \text{ or } 2R \sin A = I_2I_3 \sin \tfrac{1}{2}A.$$

Hence, $I_2I_3 = 2R \cos \tfrac{1}{2}A.$

Thus, the ***sides of the ex-central triangle*** are

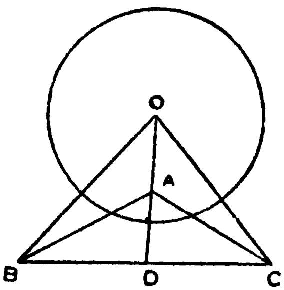
$$\mathbf{4R \cos \tfrac{1}{2}A,\ 4R \cos \tfrac{1}{2}B,\ 4R \cos \tfrac{1}{2}C}$$

34. Nine-points Circle.

The circle passing through the nine points, namely, the feet of the perpendiculars drawn from the vertices of a triangle to the opposite sides, the middle points of the sides and the middle points of the lines joining the vertices to the ortho-centre, is called the *nine-points circle* of the triangle.

It is proved in elementary Geometry that *the centre of the nine-points circle is the mid-point of the line joining the ortho-centre and the circum-centre.* It follows from the definition that *the nine-points circle is the circum-circle of the pedal triangle* and hence it follows from Art. 32 that the *radius of the nine-points circle of* $\triangle ABC$ *is* $\frac{1}{2}R$.

35. The Polar Circle.



Let ABC be a triangle in which each side is the polar of the opposite vertex with respect to a circle of centre O and radius ρ.

Then, the $\triangle ABC$ is called a *self-polar triangle* with respect to the circle ; and the circle is called the *polar circle of* $\triangle ABC$.

Let the $\angle A$ of $\triangle$ ABC be obtuse.*

From the elementary properties of poles and polars, it follows that OA, OB, OC are respectively perpendiculars to BC, and CA and AB produced ; and $\rho^2 = OA.OD$.

Hence, O *is the ortho-centre of* $\triangle ABC$,

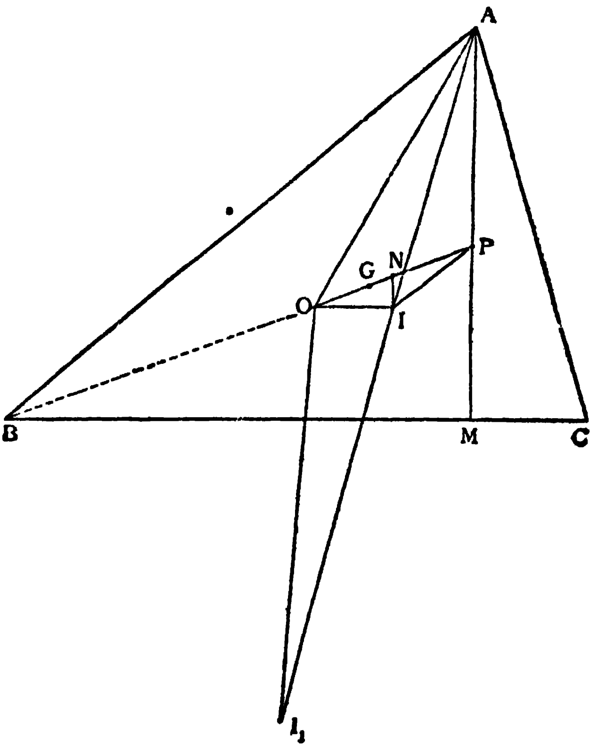
$$\text{and } \rho^2 = OA.OD = (-2R\cos A).2R\cos B\cos C.$$

[See *Art. 32, Note*]

$$\therefore\ \rho^2 = -4R^2\cos A\cos B\cos C.$$

*An acute-angled triangle in real geometry has no polar circle.

36. The distances between special points.



Let P be the ortho-centre, O the circum-centre, I the in-centre, N the nine-points centre and G the centroid of ΔABC. It is known from Geometry that the four points O, G, N, P are collinear, $ON = NP$ and $PG = 2OG$; hence $OP = 3OG$.

Also $\angle IAP = \angle IAO = \frac{1}{2}(C-B)$.* $\therefore$ $\angle OAP = C - B$.

AI_1 is the internal bisector of $\angle BAC$ and AM is perpendicular to BC.

$$AO = R \text{ ; and from Art. 32, } AP = 2R\cos A.$$

From the notes of Arts. 28 and 29, we have

$$AI = r \operatorname{cosec} \tfrac{1}{2}A = 4R \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$
$$AI_1 = r_1 \operatorname{cosec} \tfrac{1}{2}A = 4R \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

(i) *The distance between* **in-centre** *and* **ex-centre.**

$$\begin{aligned}\mathbf{I_1I} &= AI_1 - AI \\ &= 4R\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C - 4R \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \\ &= 4R \cos \tfrac{1}{2}(B+C) \\ &= \mathbf{4R \sin \tfrac{1}{2}A.}\end{aligned}$$

Similarly, $\mathbf{I_2I = 4R \sin \tfrac{1}{2}B}$; $\mathbf{I_3I = 4R \sin \tfrac{1}{2}C}$.

(ii) *The distance between* **circum-centre** *and* **ortho-centre.**

$$\begin{aligned}\mathbf{OP^2} &= OA^2 + AP^2 - 2OA.AP \cos OAP \\ &= R^2\,[1 + 4\cos^2 A - 4\cos A \cos (B-C)] \\ &= R^2[1 + 4\cos A\,\{\cos A - \cos (B-C)\}] \\ &= R^2[1 - 4\cos A\{\cos(B+C) + \cos (B-C)\}] \\ &= \mathbf{R^2\,[1 - 8\cos A \cos B \cos C].}\end{aligned}$$

(iii) *The distance between* **circum-centre** *and* **in-centre.**

$$\begin{aligned}\mathbf{OI^2} &= AO^2 + AI^2 - 2AO.AI \cos IAO \\ &= R^2[1 + 16\sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C - 8\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B-C)] \\ &= R^2[1 + 8\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C\{2\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C - \cos\tfrac{1}{2}(B-C)\}]\end{aligned}$$

*$\angle IAP = \angle IAC - \angle PAC = \frac{1}{2}A - (90° - C) = \frac{1}{2}(C-B)$,
since $90° = \frac{1}{2}(A+B+C)$.
Again, since $\angle OAB + \angle OBA + \angle AOB$, *i.e.*, $2\angle OAB + 2C = 180°$,
$\therefore$ $\angle OAB = 90° - C$.
Hence, $\angle IAO = \angle IAB - \angle OAB = \frac{1}{2}A - (90° - C) = \frac{1}{2}(C-B)$.

$$= R^2 [1 - 8 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B + C)]$$
$$= R^2 [1 - 8 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \sin \tfrac{1}{2}A]$$
$$= \mathbf{R^2 - 2Rr}, \text{ by formula (7) of Art. 28.}$$

(iv) *The distance between* **circum-centre** *and* **ex-centre.**

$$\mathbf{OI_1{}^2} = AO^2 + AI_1{}^2 - 2AO.AI_1 \cos OAI_1$$
$$= R^2 [1 + 16 \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C$$
$$- 8 \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C)]$$
$$= R^2 [1 + 8 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C], \text{ as above}$$
$$= \mathbf{R^2 + 2Rr_1}, \text{ by formula (10) Art. 29.}$$

(v) *The distance between* **in-centre** *and* **ortho-centre.**

$$IP^2 = AP^2 + AI^2 - 2AP.AI \cos IAP$$
$$= 4R^2 \cos^2 A + 16R^2 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C$$
$$- 16R^2 \cos A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C)$$
$$= 4R^2 \{\cos^2 A + 4 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C$$
$$- 4 \cos A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C)\}.$$

Now, $4 \cos A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C)$

$$= 4 \cos A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C (\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$
$$+ \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C)$$
$$= \cos A \sin B \sin C + 4 \cos A \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C ;$$

$$\therefore \quad \mathbf{IP^2} = 4R^2 [\cos^2 A + 4 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C$$
$$- \cos A \sin B \sin C - 4 \cos A \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C]$$
$$= 4R^2 [4 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C (1 - \cos A)$$
$$+ \cos A (\cos A - \sin B \sin C)]$$
$$= 4R^2 [8 \sin^2 \tfrac{1}{2}A \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C$$
$$- \cos A \{\cos (B + C) + \sin B \sin C\}]$$
$$= \mathbf{2r^2 - 4R^2 \cos A \cos B \cos C.}$$

(vi) *The distance between* **ex-centre** *and* **ortho-centre.**

Similarly, by considering $\triangle AI_1P$, it may be shown that

$$\mathbf{I_1P^2 = 2r_1{}^2 - 4R^2 \cos A \cos B \cos C.}$$

(vii) *The distance between* **in-centre** *and* **nine-points centre.**

Because N is the middle point of OP,

$\therefore$ from ΔIOP, $2(NI^2+ON^2)=IO^2+IP^2$,

or, $NI^2=\frac{1}{2}IO^2+\frac{1}{2}IP^2-\frac{1}{4}OP^2$

$=\frac{1}{2}R^2-Rr+r^2-\frac{1}{4}R^2$ [*from* (*ii*), (*iii*) & (*v*)]

$=(\frac{1}{2}R-r)^2$.

$\therefore$ $\mathbf{NI=\frac{1}{2}R-r.}$

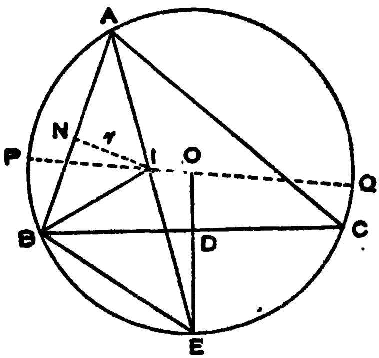
Similarly, it can be shown that,

$\mathbf{NI_1=\frac{1}{2}R+r_1.}$

Obs. Since $\frac{1}{2}R$ is the radius of the nine-points circle, the values just obtained for IN and I_1N show that the inscribed and escribed circles touch the nine-points circle. This is a trigonometrical proof of the well-known *Feuerbach's Theorem.*

Note 1. *Alternative method for OI.*

O being the circum-centre of the triangle ABC, ODE the perpendicular on BC bisects it, and meeting the circum-circle at E, it also bisects the arc BEC at E.



Hence, AE bisects the angle BAC. Thus, I, the in-centre of the triangle, lies on AE, where BI bisects the angle B. Now OI being joined and produced to meet the circum-circle at P and Q.

$AI.IE=PI.IQ=(R-OI)(R+OI)$ [$\because$ $OP=OQ=R$]

$=R^2-OI^2$.

$\therefore$ $OI^2=R^2-AI.IE$.

But $AI=r$ cosec $\frac{A}{2}$, from triangle ANI, where IN, the perpendicular from I on AB, is the in-radius r.

Again, $\angle BIE = \angle BAI + \angle ABI = \angle CAE + \angle IBC$
$$= \angle CBE + \angle IBC = \angle EBI;$$

$\therefore\ IE = EB = 2R \sin \frac{A}{2}$, since the chord BE in the circum-circle subtends an angle $BAE = \frac{A}{2}$ at the circumference.

Hence, $OI^2 = R^2 - r \operatorname{cosec} \frac{A}{2} \cdot 2R \sin \frac{A}{2} = R^2 - 2Rr.$

Note 2. In a similar way, if I_1 be an ex-centre which lies on AE, it can be proved that $OI_1{}^2 = R^2 + 2Rr_1$.

Ex. 1. *Prove that the distance of the nine-points centre from A is $\frac{1}{2}R\sqrt{1+8\cos A \sin B \sin C}$.* [*C. H. 1935.*]

With reference to the fig. of Art. 36, since N is the mid-point of OP, we have

$$2(AN^2 + PN^2) = AP^2 + AO^2,\ i.e.,\ 2AN^2 + \tfrac{1}{2}OP^2 = AP^2 + AO^2;$$

$$\therefore\ AN^2 = 2R^2 \cos^2 A + \tfrac{1}{2}R^2 - \tfrac{1}{4}(R^2 - 8R^2 \cos A \cos B \cos C)$$
$$= \tfrac{1}{4}R^2 (8\cos^2 A + 2 - 1 + 8\cos A \cos B \cos C)$$
$$= \tfrac{1}{4}R^2 [1 + 8\cos A(\cos A + \cos B \cos C)]$$
$$= \tfrac{1}{4}R^2 [1 + 8\cos A\{\cos B \cos C - \cos(B+C)\}]$$
$$= \tfrac{1}{4}R^2 [1 + 8\cos A \sin B \sin C];$$
$$\therefore\ AN = \tfrac{1}{2}R\sqrt{1 + 8\cos A \sin B \sin C}.$$

Ex. 2. *If $\sin^2 A + \sin^2 B + \sin^2 C = 1$, show that the circum-circle cuts the nine-points circle orthogonally.* [*C. H. 1935.*]

When two circles cut orthogonally, the square of the distance between their centres is equal to the sum of the squares of their radii.

Hence, if the circum-circle and the nine-points circle cut orthogonally, then

$$ON^2 = R^2 + (\tfrac{1}{2}R)^2 = \tfrac{5}{4}R^2. \qquad \cdots \qquad \cdots \quad (1)$$

Again, $ON^2 = \frac{1}{4}OP^2 = \frac{1}{4}R^2(1 - 8\cos A \cos B \cos C)$. $\cdots$ (2)

Since, $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$,

hence, the required result follows.

Ex. 8. *Prove that the in-circle will pass through the ortho-centre if* $2 \cos A \cos B \cos C = (1-\cos A)(1-\cos B)(1-\cos C)$.

Since the in-circle passes through the ortho-centre, $\therefore$ $IP = r$.

$$\therefore\ r^2 = IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

$$\therefore\ 4R^2 \cos A \cos B \cos C = r^2$$
$$= 16R^2 \sin^2 \tfrac{1}{2}A \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C$$
$$= 2R^2 (1-\cos A)(1-\cos B)(1-\cos C).$$

Hence, the result.

EXAMPLES IV(b)

1. If l, m, n be the lengths of the medians of a triangle, prove that

$$(b^2-c^2)l^2+(c^2-a^2)m^2+(a^2-b^2)n^2=0.$$

2. If x, y, z be the lengths of the internal bisectors of the angles of a triangle and l, m, n be the lengths of these bisectors produced to meet the circum-circle, show that

(i) $x^{-1} \cos \frac{1}{2}A + y^{-1} \cos \frac{1}{2}B + z^{-1} \cos \frac{1}{2}C = a^{-1}+b^{-1}+c^{-1}$.

(ii) $l \cos \frac{1}{2}A + m \cos \frac{1}{2}B + n \cos \frac{1}{2}C = a+b+c$.

3. (i) If the internal bisectors of the angles of a triangle make angles α, β, γ with the sides a, b, c, show that

$$a \sin 2\alpha + b \sin 2\beta + c \sin 2\gamma = 0.$$

[$\alpha = \frac{1}{2}A + B = 90° + \frac{1}{2}B - \frac{1}{2}C$.]

(ii) If the lengths of the internal bisectors of the two angles of a triangle be equal, show that the triangle is isosceles.

4. If G be the centroid of the triangle ABC and x, y, z be the circum-radii of the triangles BGC, CGA, AGB, show that

$$\frac{a^2(b^2-c^2)}{x^2}+\frac{b^2(c^2-a^2)}{y^2}+\frac{c^2(a^2-b^2)}{z^2}=0.$$

[*C. H. 1950.*]

5. If f, g, h denote the sides of the pedal triangle, prove that

$$\frac{(b^2-c^2)f}{a^2}+\frac{(c^2-a^2)g}{b^2}+\frac{(a^2-b^2)h}{c^2}=0.$$

6. Prove that the in-radius of the pedal triangle is

$$2R \cos A \cos B \cos C.$$

7. If LMN be the pedal triangle and P the orthocentre, prove that

(i) $\dfrac{PL}{AL}+\dfrac{PM}{BM}+\dfrac{PN}{CN}=1,$

(ii) $\dfrac{yz}{bc}+\dfrac{zx}{ca}+\dfrac{xy}{ab}=\dfrac{1}{4},$

where x, y, z are the circum-radii of the triangles MPN, NPL, LPM respectively.

8. If L, M, N be the projections of the vertices of the triangle ABC on the opposite sides, show that

(i) the area of the triangle LMN

$$=2\Delta \cos A \cos B \cos C$$
$$=\tfrac{1}{2}R^2 \sin 2A \sin 2B \sin 2C.$$

(ii) the perimeter of the triangle LMN

$$=4R \sin A \sin B \sin C$$
$$=\frac{\Delta}{R}.$$

9. Prove that the areas of the triangles $I_1I_2I_3$, I_2I_3I, I_3I_1I, I_1I_2I are inversely as r, r_1, r_2, r_3; and the circum-radius of each of these triangles is $2R$. [*C. H. 1937.*]

10. Show that the area of the triangle formed by joining the centres of the escribed circle is

$$\frac{abc}{2r}, \quad \text{or,} \quad 8R^2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

11. Prove that (letters having same significance as in Art. 36) ;

(i) (*a*) area of $\triangle IOP$

$$= -2R^2 \sin \tfrac{1}{2}(B-C) \sin \tfrac{1}{2}(C-A) \sin \tfrac{1}{2}(A-B).$$

(*b*) area of $\triangle IPG$

$$= -\tfrac{4}{3}R^2 \sin \tfrac{1}{2}(B-C) \sin \tfrac{1}{2}(C-A) \sin \tfrac{1}{2}(A-B).$$

(ii) $OG^2 = R^2 - \frac{1}{9}(a^2+b^2+c^2)$.

(iii) $\cos A + \cos B + \cos C = \frac{3}{2}$, if $\triangle IOP$ be equilateral.

(iv) If $OI = OP$, prove that $\Sigma \cos A + \Sigma \cos 2A = 0$.

12. If t_1, t_2, t_3 be the lengths of the tangents from the ex-centres to the circum-circle, prove that

(i) $\dfrac{1}{t_1{}^2} + \dfrac{1}{t_2{}^2} + \dfrac{1}{t_3{}^2} = \dfrac{a+b+c}{abc}$. [*C. H. 1934.*]

(ii) $t_1 t_2 t_3 = abc\sqrt{\dfrac{R}{2r}}$.

13. If x, y, z be the distances of the ex-centres of a triangle from the in-centre and d the diameter of the circum-circle, show that

$$xyz + d(x^2+y^2+z^2) = 4d^3.$$ [*C. H. 1933.*]

14. If x, y, z be the distances of the nine-points centre from the vertices of a triangle and k its distance from the ortho-centre, prove that

$$x^2+y^2+z^2+k^2 = 3R^2.$$ [*C. H. 1939.*]

15. The perpendiculars from the angular points of a triangle on the straight line joining the in-centre and the ortho-centre are p, q, r ; prove that

$$\frac{p \sin A}{\sec B - \sec C} = \frac{q \sin B}{\sec C - \sec A} = \frac{r \sin C}{\sec A - \sec B}.$$

16. If the in-centre be equidistant from the circum-centre and the ortho-centre, prove that one angle of the triangle is 60°. [*C. H. 1931.*]

17. If I be the in-centre of the triangle ABC, ID, IE, IF be the perpendiculars on the sides ρ_1, ρ_2, ρ_3, the radii of the circles inscribed in the quadrilaterals $AEIF$, $BFID$, $CDIE$, prove that

(i) $\dfrac{\rho_1}{r-\rho_1}+\dfrac{\rho_2}{r-\rho_2}+\dfrac{\rho_3}{r-\rho_3}=\dfrac{s}{r}$.

(ii) $\Sigma\left(\dfrac{1}{\rho_2}-\dfrac{1}{r}\right)\left(\dfrac{1}{\rho_3}-\dfrac{1}{r}\right)=\dfrac{1}{r^2}$

18. If x, y, z be the lengths of the three internal bisectors of the angles of a triangle, prove that

(i) $(b+c)^2\dfrac{x^2}{bc}+(c+a)^2\dfrac{y^2}{ca}+(a+b)^2\dfrac{z^2}{ab}=(a+b+c)^2$.

(ii) $4\Delta=\left\{\dfrac{xyz}{2Rs}(b+c)(c+a)(a+b)\right\}^{\frac{1}{2}}$.

19. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of a given triangle whose area is Δ, and Δ_1, Δ_2, Δ_3, the corresponding areas for the escribed circles, prove that

(i) $\dfrac{\Delta_0}{r}=\dfrac{\Delta_1}{r_1}=\dfrac{\Delta_2}{r_2}=\dfrac{\Delta_3}{r_3}$.

(ii) $\Delta_1+\Delta_2+\Delta_3-\Delta_0=2\Delta$.

20. Inside a triangle ABC, lines AO, BO, CO are drawn such that $\angle OAB=\angle OBC=\angle OCA=\omega$.

[*O is called the Brocard Point.*]

Prove that

(i) $\sin^2\omega=\sin(A-\omega)\sin(B-\omega)\sin(C-\omega$

(ii) $\cot \omega = \cot A + \cot B + \cot C$.

(iii) $\operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$.

21. Perpendiculars AL, BM, CN are drawn from the angles A, B, C of an acute-angled triangle on the opposite sides and produced to meet the circum-circle in L', M', N'; if LL', MM', NN' be α, β, γ respectively, show that

(i) $\dfrac{a}{\alpha} + \dfrac{b}{\beta} + \dfrac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$.

(ii) $\dfrac{AL'}{AL} + \dfrac{BM'}{BM} + \dfrac{CN'}{CN} = 4$.

(iii) $\Delta L'M'N' = 8\Delta \cos A \cos B \cos C$.

22. If the circum-centre lies on the in-circle, prove that
$$\cos A + \cos B + \cos C = \sqrt{2}.$$
[*C. P. 1951.*]

23. If $R = 2r$, show that the triangle is equilateral.
[*C. P. 1940.*]

24. A circle is inscribed in an equilateral triangle; an equilateral triangle in the circle, a circle again in the latter triangle and so on; in this way $(n+1)$ circles are described; if $r, x_1, x_2, \ldots x_n$ be the radii of the circles, show that

$$r = x_1 + x_2 + x_3 + \cdots\cdots + x_{n-1} + 2x_n.$$

[*For an equilateral triangle $R = 2r$.*]

25. If the distance between the ortho-centre and the circum-centre is $\frac{1}{2}a$, show that the triangle is right-angled, or else, $\tan B \tan C = 9$. [*C. H. 1933* ; *P. P. 1931*]

26. If I be the in-centre and x, y, z the circum-radii of triangles BIC, CIA, AIB, then show that $xyz = 2R^2 r$.

27. If the in-circle touches the sides of the triangle ABC in A_1, B_1, C_1, and if ρ_1, ρ_2, ρ_3 are the circum-radii of the triangles B_1IC_1, C_1IA_1, A_1IB_1, prove that $2\rho_1\rho_2\rho_3 = Rr^2$.

28. The circle inscribed in the triangle ABC touches the sides BC, CA, AB at the points A_1, B_1, C_1 respectively; similarly, the circle inscribed in the triangle $A_1B_1C_1$ touches the sides at A_2, B_2, C_2 respectively and so on; if $A_nB_nC_n$ be the nth triangle so formed, find its angles and show that this triangle ultimately becomes equilateral.

29. A triangle is formed by joining the mid-points of the sides of a given triangle; another by joining the mid-points of the sides of the new triangle and so on. If r, r_n be the in-radii of the original and the nth of the new triangles respectively. then will $r_n = \frac{1}{2^n} r$; and if R, R_n be the radii of circum-circles, then will $R_n = \frac{1}{2^n} R$.

30. If x, y, z are respectively equal to IA, IB, IC and x_1, y_1, z_1 are respectively equal to I_1A, I_2B, I_3C, then show that (i) $\dfrac{b-c}{ax_1^2} + \dfrac{c-a}{by_1^2} + \dfrac{a-b}{cz_1^2} = 0.$

(ii) $x^2\left(\dfrac{1}{b} - \dfrac{1}{c}\right) + y^2\left(\dfrac{1}{c} - \dfrac{1}{a}\right) + z^2\left(\dfrac{1}{a} - \dfrac{1}{b}\right) = 0.$

[See *Notes of Arts. 28 and 29.*]

31. Prove that the nine-points circle does not cut the circum-circle unless the triangle is obtuse-angled.

[*C. H. 1936.*]

32. If the escribed circle which touches the side a of a triangle is equal to the circum-circle, prove that

$\cos A = \cos B + \cos C.$ [*C. P. 1945, '47.*]

ANSWERS

28. $A_n = \dfrac{\pi}{3}\left\{1 + (-1)^{n+1}\dfrac{1}{2^n}\right\} + (-1)^n \dfrac{A}{2^n}$

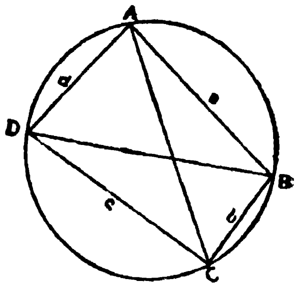
$B_n = \dfrac{\pi}{3}\left\{1 + (-1)^{n+1}\dfrac{1}{2^n}\right\} + (-1)^n \dfrac{B}{2^n}$

$C_n = \dfrac{\pi}{3}\left\{1 + (-1)^{n+1}\dfrac{1}{2^n}\right\} + (-1)^n \dfrac{C}{2^n}.$

CHAPTER V

PROPERTIES OF QUADRILATERALS

37. Area of a cyclic quadrilateral.



Let $ABCD$ be a cyclic quadrilateral and let $AB=a$, $BC=b$, $CD=c$, $DA=d$.

We know from Geometry,

$B+D=180°$,

i.e., $D=180°-B$ and hence,

$$\sin D=\sin(180°-B)$$
$$=\sin B\,;$$

and $\cos D=\cos(180°-B)$
$$=-\cos B.$$

Now, quad. $ABCD=\triangle ABC+\triangle ADC$

$$=\tfrac{1}{2}ab\sin B+\tfrac{1}{2}cd\sin D=\tfrac{1}{2}(ab+cd)\sin B. \quad \cdots \quad (1)$$

$[\because \ \sin D=\sin B]$

Again, from $\triangle ABC$, $AC^2=a^2+b^2-2ab\cos B$

and from $\triangle ACD$, $AC^2=c^2+d^2-2cd\cos D$

$$=c^2+d^2+2cd\cos B.$$

$[\because \cos D=-\cos B]$

$$\therefore \quad c^2+d^2+2cd\cos B=a^2+b^2-2ab\cos B.$$

$$\cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}. \quad \cdots \quad (2)$$

Hence, $\sin^2 B=1-\cos^2 B=1-\left\{\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$

$$=\frac{\{2(ab+cd)\}^2-(a^2+b^2-c^2-d^2)^2}{4(ab+cd)^2}$$

$$=\frac{\{2(ab+cd)+(a^2+b^2-c^2-d^2)\}\{2(ab+cd)-(a^2+b^2-c^2-d^2)\}}{4(ab+cd)^2}$$

$$=\frac{\{(a^2+b^2+2ab)-(c^2+d^2-2cd)\}\{(c^2+d^2+2cd)-(a^2+b^2-2ab)\}}{4(ab+cd)^2}$$

$$=\frac{\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}}{4(ab+cd)^2}$$

$$=\frac{(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)}{4(ab+cd)^2}.$$

Let $a+b+c+d=2s$; then

$$a+b+c-d=a+b+c+d-2d=2(s-d).$$

Similarly, $a+b+d-c=2(s-c)$; $a+c+d-b=2(s-b)$, and $b+c+d-a=2(s-a)$.

Hence, $\sin^2 B=\dfrac{16(s-a)(s-b)(s-c)(s-d)}{4(ab+cd)^2}$.

$$\therefore \quad \sin B=\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ab+cd)}. \quad \cdots \quad (3)$$

Substituting this value of $\sin B$ in (1), we have the area of the quadrilateral $ABCD$

$$=\sqrt{(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})(\mathbf{s}-\mathbf{d})}.$$

Note. The expression for the area of a quadrilateral was discovered by ***Brahmagupta,*** a Hindu Mathematician of the sixth century.

38. *The diagonals and circum-radius of the cyclic quadrilateral.*

Let x, y be the lengths of the diagonals AC, BD of the cyclic quadrilateral $ABCD$.

From the above figure, (fig. of Art. 37)

$$AC^2=a^2+b^2-2ab\cos B$$

$$=a^2+b^2-2ab\cdot\frac{a^2+b^2-c^2-d^2}{2(ab+cd)},$$

on substituting the value of $\cos B$ from (2),

$$=\frac{(a^2+b^2)cd+(c^2+d^2)ab}{ab+cd}. \qquad \cdots \quad (4)$$

$$\therefore \quad \mathbf{x^2=\frac{(ac+bd)(ad+bc)}{ab+cd}}. \qquad \cdots \quad (5)$$

Similarly, it may be shown that

$$\cos A=\frac{a^2+d^2-b^2-c^2}{2(ad+bc)},$$

$$\text{and} \quad \mathbf{y^2}=BD^2=\mathbf{\frac{(ac+bd)(ab+cd)}{ad+bc}}. \qquad \cdots \quad (6)$$

From (5) and (6), $x^2y^2=(ac+bd)^2$.

$$\therefore \quad \mathbf{xy=ac+bd}. \qquad \cdots \quad (7)$$

This is known as **Ptolemy's* Theorem.**

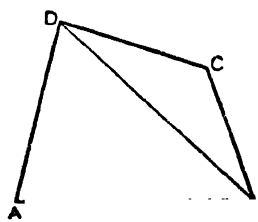
Let ρ be the radius of the circle circumscribing the quadrilateral $ABCD$; then it is the same as the circum-radius of the triangle ABC ; hence

$$\rho=\frac{AC}{2\sin B}=\frac{1}{4}\left\{\frac{(\mathbf{ab+cd})(\mathbf{ac+bd})\,(\mathbf{ad+bc})}{(\mathbf{s-a})(\mathbf{s-b})(\mathbf{s-c})(\mathbf{s-d})}\right\}^{\frac{1}{2}}, \qquad \cdots \quad (8)$$

on substituting the values of AC and $\sin B$.

39. Area of any quadrilateral.

Let S denote the area of the quadrilateral $ABCD$ and let $AB=a$, $BC=b$, $CD=c$, $DA=d$.



By equating the two values of BD^2 found from the Δs BAD, BCD, we have

$$a^2+d^2-2ad\cos A=b^2+c^2-2bc\cos C.$$

$$\therefore \quad a^2+d^2-b^2-c^2=2ad\cos A-2bc\cos C. \qquad \cdots \quad (9)$$

*Ptolemy, a celebrated astronomer, who flourished in 139 A. D. did many researches in Astronomy, Trigonometry and Geometry.

Now, $S = \triangle ABD + \triangle BCD$

$$= \tfrac{1}{2}ad \sin A + \tfrac{1}{2}bc \sin C.$$

$$\therefore \quad 4S = 2ad \sin A + 2bc \sin C. \quad \ldots \quad \ldots (10)$$

Squaring (9) and (10) and adding,

$$16S^2 + (a^2 + d^2 - b^2 - c^2)^2$$
$$= 4a^2d^2 + 4b^2c^2 - 8abcd \cos (A + C). \quad \ldots (11)$$

Let $A + C = 2\alpha$,

then $\cos (A + C) = \cos 2\alpha = 2 \cos^2\alpha - 1.$

$\therefore$ the right side of (11)

$$= 4a^2d^2 + 4b^2c^2 + 8abcd - 16abcd \cos^2\alpha$$
$$= 4(ad + bc)^2 - 16abcd \cos^2\alpha.$$

Hence, from (11), we have

$$16S^2 = 4(ad + bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd \cos^2\alpha.$$

Now, as in Art. 36, it can be easily shown that

$$4(ad + bc)^2 - (a^2 + d^2 - b^2 - c^2)^2$$
$$= 16(s-a)(s-b)(s-c)(s-d);$$

$$\therefore \quad \mathbf{S = [(s-a)(s-b)(s-c)(s-d) - abcd \cos^2\alpha]^{\frac{1}{2}}}. \quad \ldots (12)$$

Cor. This expression for the area shows that *the quadrilateral of which the sides are given, has its area greatest* when $\alpha = \frac{1}{2}\pi$, that is where 2α or $A + C = \pi$; in other words *when the quadrilateral can be inscribed in a circle (i.e., is cyclic).*

Note. On putting $d = 0$ in the formula (12), we get the expression for the area of a triangle, since in this case the quadrilateral reduces to the triangle ABC.

Ex. 1. *Find the area of a quadrilateral which can be inscribed in one and circumscribed about another circle.* [*C. P. 1927.*]

If the circle inscribed in the quadrilateral $ABCD$ touches the sides AB, BC, CD, DA at the points E, F, G, H, then, we have

$$AE = AH, \; BE = BF, \; CF = CG, \; DG = DH.$$

$\therefore$ $AE + BE + CG + DG = AH + BF + CF + DH,$

i.e., $AB + CD = BC + AD$, or, $a + c = b + d.$

Hence, $s=\frac{1}{2}(a+b+c+d)=a+c=b+d.$

$\therefore$ $s-a=c,\ s-b=d,\ s-c=a,\ s-d=b.$ ... (1)

By Art. 37, the area of the quadrilateral which can be inscribed in a circle is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Hence, substituting the values of $s-a$, $s-b$, $s-c$, $s-d$ from (1) the required area is equal to $\sqrt{abcd}$.

Ex. 2. *ABCD is a cyclic quadrilateral, the circle having unit radius ; α, β, γ being the angles subtended by AB, BC, CD at the circumference, prove that*

$$\text{area of } ABCD=2\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta).$$

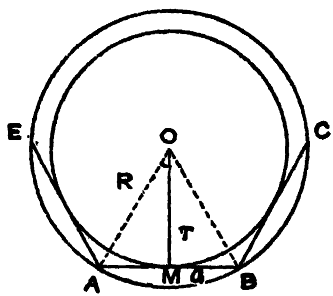
O being the centre of the circum-circle, $\angle AOB=2\alpha$, $\angle BOC=2\beta$, $\angle COD=2\gamma$ and so $\angle AOD=2(\pi-\alpha-\beta-\gamma)$.

Now, area of $ABCD=\triangle AOB+\triangle BOC+\triangle COD+\triangle DOA$

$$=\tfrac{1}{2}\{\sin 2\alpha+\sin 2\beta+\sin 2\gamma+\sin 2(\pi-\alpha-\beta-\gamma)\}$$
$$[\because AO=BO=CO=DO=1 \text{ here.}]$$
$$=\tfrac{1}{2}\{\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin 2(\alpha+\beta+\gamma)\}$$
$$=\sin(\alpha+\beta)\cos(\alpha-\beta)-\sin(\alpha+\beta)\cos(\alpha+\beta+2\gamma)$$
$$=2\sin(\alpha+\beta)\sin(\alpha+\gamma)\sin(\beta+\gamma).$$

40. Circum-radius, In-radius and area of a regular Polygon.

(i) Let O be the centre and R the radius of the circle circumscribed about a regular polygon of n sides, of which $AB(=a)$ is one side. Join OA, OB.



Draw OM perp. to AB ;

then $AM=BM=\frac{a}{2}$.

$$\angle AOM=\tfrac{1}{2}\angle AOB=\frac{1}{2}\cdot\frac{2\pi}{n}=\frac{\pi}{n}.$$

$\therefore$ $\mathbf{R}=OA=AM\operatorname{cosec} AOM=\frac{\mathbf{a}}{\mathbf{2}}\operatorname{\mathbf{cosec}}\frac{\boldsymbol{\pi}}{\mathbf{n}}.$ (1)

(ii) O being the centre and r the radius of the circle inscribed in the regular polygon of n sides, of which AB $(=a)$ is one side,

$$AM = BM = \frac{a}{2},$$

and $\angle AOM = \frac{1}{2}\angle AOB = \frac{\pi}{n}$ as before.

$$\mathbf{r} = OM = AM \cot AOM = \frac{\mathbf{a}}{\mathbf{2}} \cot \frac{\boldsymbol{\pi}}{\mathbf{n}}. \quad \dots \quad (2)$$

(iii) **Area** of the polygon $= n \times$ area of $\triangle AOB$

$$= n.\tfrac{1}{2}AB.OM = n.\tfrac{1}{2}a.\frac{a}{2}\cot\frac{\pi}{n}$$

$$= \frac{\mathbf{na^2}}{\mathbf{4}} \cot \frac{\boldsymbol{\pi}}{\mathbf{n}}. \quad \dots \quad \dots \quad (3)$$

Substituting the value of a from (1) and (2) in (3),

$$\text{area is also} = \frac{n}{2}R^2 \sin\frac{2\pi}{n} \quad \dots \quad (4)$$

$$\text{and} \quad = nr^2 \tan\frac{\pi}{n}. \quad \dots \quad \dots \quad (5)$$

EXAMPLES V

1. Show that the area of a quadrilateral is equal to half the product of the two diagonals and the sine of the angle between them.

2. If x, y be the diagonals of a quadrilateral and α the angle between them, show that

$$2xy \cos \alpha = (a^2 + c^2) \sim (b^2 + d^2).$$

3. Show that the area of a quadrilateral whose diagonals are x, y is

$$\tfrac{1}{4}\{4x^2y^2 - (a^2 + c^2 - b^2 - d^2)^2\}^{\frac{1}{2}}. \quad [\textit{P. H. 1935.}]$$

4. If θ be the angle between the diagonals of a quadrilateral, show that the area is

$$\tfrac{1}{2}\{(a^2 + c^2) \sim (b^2 + d^2)\} \tan \theta.$$

5. If ρ be the radius of the circle which can be inscribed in a cyclic quadrilateral, show that

$$\rho = \frac{2\sqrt{abcd}}{a+b+c+d} = \left\{\frac{abcd}{(a+c)(b+d)}\right\}^{\frac{1}{2}}. \quad [\ P.\ H.\ 1934.\ ]$$

6. Show that if a quadrilateral of given sides be such that a circle can be inscribed in it, the circle is the greatest when the quadrilateral can be inscribed in a circle.

7. If a quadrilateral circumscribes a circle, prove that

$$4S^2 = x^2y^2 - (ac - bd)^2.$$

8. If AB, BC, CD are three sides of a quadrilateral of lengths a, b, c respectively, and if $\angle ABC = \alpha$, $\angle BCD = \beta$ and the angle between AB and DC produced is γ, prove that

$$AD^2 = a^2 + b^2 + c^2 - 2ab \cos \alpha - 2bc \cos \beta - 2ca \cos \gamma.$$

9. Show that the area of a quadrilateral which is circumscribed about a circle is $\sqrt{abcd} \sin \alpha$, where 2α is the sum of a pair of opposite angles.

10. If θ be the angle between the diagonals of a cyclic quadrilateral, show that

$$\sin \theta = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ac+bd}.$$

If the same quadrilateral can also be circumscribed about a circle, prove that

$$\cos \theta = \frac{ac-bd}{ac+bd}.$$

11. If the diagonals of a cyclic quadrilateral $ABCD$ intersect at O, show that

$$\frac{OA}{DA.AB} = \frac{OB}{AB.BC} = \frac{OC}{BC.CD} = \frac{OD}{CD.DA}.$$

12. If ρ be the radius of the circle inscribed in a quadrilateral whose area is S, then

$$\rho = \frac{S}{s} = \frac{S}{a+c} = \frac{S}{b+d} = \frac{S}{\{(a+c)(b+d)\}^{\frac{1}{2}}}.$$

13. If the quadrilateral $ABCD$ is circumscribable, prove that $\qquad \sqrt{ab} \sin \frac{1}{2}B = \sqrt{cd} \sin \frac{1}{2}D.$

14. A diagonal of a quadrilateral makes angles α, β with the sides at one of its ends and angles γ, δ with the sides at the other end, the angles α, γ being on the same side of the diagonal. If θ be the angle between the diagonals, show that

$$\tan\theta = \frac{\cot\alpha + \cot\beta + \cot\gamma + \cot\delta}{\cot\alpha\cot\delta \sim \cot\beta\cot\gamma}.$$

15. A quadrilateral is such that it can be inscribed in one and circumscribed about another circle. If R and r be the circum-radius and in-radius of the quadrilateral, and x the distance between the centres of these circles, show that

$$\frac{1}{r^2} = \frac{1}{(R+x)^2} + \frac{1}{(R-x)^2}.$$

16. A quadrilateral is formed of four jointed rods of lengths a, b, c, d. If the area of the quadrilateral when the angle between a, b is a right angle is equal to the area when the angle between c, d is a right angle, show that

$$ab = cd, \text{ or, } a^2 + b^2 = c^2 + d^2.$$

17. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

[*C. P. 1941.*]

18. Prove that the areas of two regular polygons of n sides and $2n$ sides and of equal perimeter are as $\cos\frac{\pi}{n} : \cos^2\frac{\pi}{2n}$.

19. (i) Two regular polygons of n sides are respectively circumscribed about and inscribed in a circle. Prove that their areas are as $1 : \cos^2\frac{\pi}{n}$.

(ii) If p_1, p_2, p_3 be the perimeters of the circumscribing polygon, the circle, and the inscribed polygon, then

$$p_1 : p_2 : p_3 = \sec\frac{\pi}{n} : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} : 1.$$

20. If r_n and R_n denote the in-radius and circum-radius of a regular n-gon of given perimeter, prove that

(i) $2r_{2n} = r_n + R_n$.
(ii) $R^2_{2n} = R_n \cdot r_{2n}$.

MISCELLANEOUS EXAMPLES I

Solve (*Ex. 1 to 9*) :—

1. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$

2. $2 \cos x \cos y = 1,\ \tan x + \tan y = 2.$

3. $\cos 3x + 3 \sin 2x = 3 \cos x.$

4. $\cos^3 a \sec x + \sin^3 a \operatorname{cosec} x = 1.$

5. $\tan x + \tan (x + a) + \tan (x + \beta)$
$$= \tan x \tan (x + a) \tan (x + \beta).$$

6. $\tan (x + b) \tan (x + c) + \tan (x + c) \tan (x + a)$
$$+ \tan (x + a) \tan (x + b) = 1.$$

7. $\tan \theta = \sin 3\phi,\ \sin \theta = \tan 3\phi.$

8. $\cot \theta \tan 2\theta - \tan \theta \cot 2\theta = 2.$

9. $p \sin^4\theta - q \sin^4\phi = p.$

$p \cos^4\theta - q \cos^4\phi = q.$

10. If $\tan ax - \tan bx = 0$, show that the values of x form a series in A. P.

Prove the following relations (*Ex. 11 to 16*) :—

11. $\tan^{-1}\left(\dfrac{a \cos \theta}{1 - a \sin \theta}\right) - \cot^{-1}\left(\dfrac{\cos \theta}{a - \sin \theta}\right) = 0.$

12. $\tan^{-1}\dfrac{3 \sin 2\theta}{5 + 3 \cos 2\theta} + \tan^{-1}\left(\dfrac{1}{4} \tan \theta\right) = \theta.$ [*C. P. 1949.*]

13. $\tan^{-1}\dfrac{\tan x + \tan y}{1 + \tan x \tan y} = \frac{1}{2} \sin^{-1}\dfrac{\sin 2x + \sin 2y}{1 + \sin 2x \sin 2y}.$

14. $\cos^{-1}\dfrac{\cos x + \cos y}{1 + \cos x \cos y} = 2 \tan^{-1}\left(\tan \dfrac{x}{2} \tan \dfrac{y}{2}\right).$

[*C. P. 1946.*]

15. $$\frac{x^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{x}{y}\right)+\frac{y^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{y}{x}\right)=(x+y)(x^2+y^2).$$

16. $2\cot^{-1}5+\cot^{-1}7+2\cot^{-1}8=\frac{1}{4}\pi.$

17. If $\sin^2 x+\sin^2 y=\frac{1}{2}$, then show that $(2n+1)\frac{1}{2}\pi$ is one of the values of z which satisfy the equation

$$z=\sin^{-1}(\sin x+\sin y)+\sin^{-1}(\sin x-\sin y).$$

18. If $\tan^{-1}p+\tan^{-1}q+\tan^{-1}r=\frac{1}{2}\pi$, show that

$$qr+rp+pq=1.$$

19. If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$, then show that

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta.$$ [*C. H. 1943.*]

20. Solve : (i) $\operatorname{cosec}^{-1}x=\operatorname{cosec}^{-1}a+\operatorname{cosec}^{-1}b.$

(ii) $\cos^{-1}x+\cos^{-1}2x=\frac{1}{3}\pi.$

[*In the following examples in the properties of a triangle, letters have the same significance as in the preceding Articles.*]

21. If x, y, z are the distances of the ortho-centre from A, B, C respectively, show that $a^2+x^2=b^2+y^2=c^2+z^2$.

22. If p, q, r denote the sides of the $\Delta I_1I_2I_3$, then show that $\frac{a^2}{p^2}+\frac{b^2}{q^2}+\frac{c^2}{r^2}+\frac{2abc}{pqr}=1.$

23. Show that in a triangle

$$2bc+2ca+2ab-a^2-b^2-c^2=4r^2+16Rr.$$

24. If x, y, z be the ratios which the sides a, b, c of a triangle bear to the perpendiculars on them from the opposite angles A, B, C, then $\Sigma x^2-2\Sigma yz+4=0$.

25. Prove that the in-radius of $\triangle\, I_1 I_2 I_3$ is
$$2R\,\{\sin \tfrac{1}{2}A + \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C - 1\}.$$

26. If IG be parallel to BC, show that $r_1 = 3r$.

27. Prove that

(i) $AN^2 = \frac{1}{4}(R^2 + b^2 + c^2 - a^2)$.
(ii) $OP^2 = 9R^2 - a^2 - b^2 - c^2$.
(iii) $OI^2 = R^2\,[3 - 2\Sigma \cos A]$.
(iv) $OI^2 + OI_1{}^2 + OI_2{}^2 + OI_3{}^2 = 12R^2$.
(v) $AI^2 + AI_1{}^2 + AI_2{}^2 + AI_3{}^2 = 16R^2$.
(vi) $NA^2 + NB^2 + NC^2 = R^2\,(\frac{11}{4} + 2\cos A \cos B \cos C)$.
(vii) $NI + NI_1 + NI_2 + NI_3 = 6R$.
(viii) $AG^2 + BG^2 + CG^2 = \frac{8}{3}R^2\,(1 + \cos A \cos B \cos C)$.

28. If $AP = r$, prove that the circum-circle cuts the escribed circle opposite to A orthogonally.

29. Prove that $\dfrac{2\triangle ABC}{\triangle I_1 I_2 I_3} = \dfrac{a\cos A + b\cos B + c\cos C}{a + b + c}$.

30. The internal bisectors of the angles of a triangle ABC meet the opposite sides in D, E, F; show that the area of $\triangle DEF = \dfrac{2\triangle abc}{(b+c)(c+a)(a+b)}$.

31. DEF is the triangle formed by joining the points of contact of the in-circle with the sides of $\triangle ABC$; prove that

(i) its sides are $2r\cos\frac{1}{2}A$, $2r\cos\frac{1}{2}B$, $2r\cos\frac{1}{2}C$.
(ii) its angles are $\frac{1}{2}\pi - \frac{1}{2}A$, $\frac{1}{2}\pi - \frac{1}{2}B$, $\frac{1}{2}\pi - \frac{1}{2}C$.
(iii) its area is $\dfrac{1}{2}\,\dfrac{r}{R}\,\triangle$.

32. If $\rho, \rho_1, \rho_2, \rho_3$ are the in-radius and ex-radii of the pedal triangle, then
$$\frac{\rho_1\rho_2\rho_3}{\rho} = \frac{rr_1r_2r_3}{R^2}.$$

33. A straight line AD drawn through A meets the base BC in D;

if $BD : CD = m : n$ and if $\angle BAD = \alpha$ and $\angle CAD = \beta$, and if $\angle ADB = \theta$, then show that

(i) $(m+n) \cot \theta = n \cot \beta - m \cot \alpha$.

(ii) $(m+n) \cot \theta = m \cot C - n \cot B$.

(iii) If D be the middle point of BC, show that

$$\cot \alpha - \cot \beta = \cot B - \cot C. \quad [\textit{ C. H. 1936.}]$$

34. If the medians of a triangle meet the opposite sides in D, E, F, and if G be the centroid and if the angles BAD, CBE, ACF are θ, ϕ, ψ and the angles CAD, ABE, BCF are θ', ϕ', ψ', prove that $\Sigma \cot \theta = \Sigma \cot \theta'$.

35. If a, b, c, d be the lengths of the four sides of a quadrilateral and if θ be the angle between their diagonals and if $a^2 + c^2 = b^2 + d^2$, show that $\theta = 90°$.

36. If p_1, p_2, p_3, p_4 be the perpendiculars from the angles of a quadrilateral upon the diagonals x, y, and if θ be the angle between the diagonals, show that

$$\sin \theta = \left\{\frac{(p_1 + p_3)(p_2 + p_4)}{xy}\right\}^{\frac{1}{2}}. \quad [\textit{ P. H. 1933.}]$$

37. If $ABCD$ be a cyclic quadrilateral, prove that

$$\tan^2 \frac{B}{2} = \frac{(s-a)(s-b)}{(s-c)(s-d)}.$$

38. If θ be the angle between the diagonals of a cyclic quadrilateral, prove that

$$\tan^2 \frac{\theta}{2} = \frac{(s-b)(s-d)}{(s-a)(s-c)}, \text{ or, } \frac{(s-a)(s-c)}{(s-b)(s-d)}.$$

ANSWERS

1. $(2n+1)\frac{\pi}{2}$ or $(2n+1)\frac{\pi}{4}$ or $(2n+1)\frac{\pi}{8}$. 2. $mx+\frac{\pi}{4}$, $(2k-m)\pi-\frac{\pi}{4}$.

3. $n\pi+\frac{\pi}{2}$, $n\pi+(-1)^n\frac{\pi}{6}$.

4. $2n\pi+a$ and $n\pi-a+(-1)^{n-1}\sin^{-1}(\sin a\cos a)$.

5. $\frac{1}{3}(n\pi-a-\beta)$. 6. $\frac{1}{3}\{n\pi+\frac{1}{2}\pi-(a+b+c)\}$. 7. $\theta=n\pi$, $\phi=\frac{1}{3}m\pi$.

8. $(2n+1)\frac{\pi}{8}$. 9. $\theta=\cos^{-1}\left(\frac{q^2}{pq-p^2}\right)^{\frac{1}{4}}$; $\phi=\cos^{-1}\left(\frac{p}{q-p}\right)^{\frac{1}{4}}$.

10. (i) $\frac{ab}{\sqrt{a^2-1}+\sqrt{b^2-1}}$. (ii) $\pm\frac{1}{2}$.

CHAPTER VI

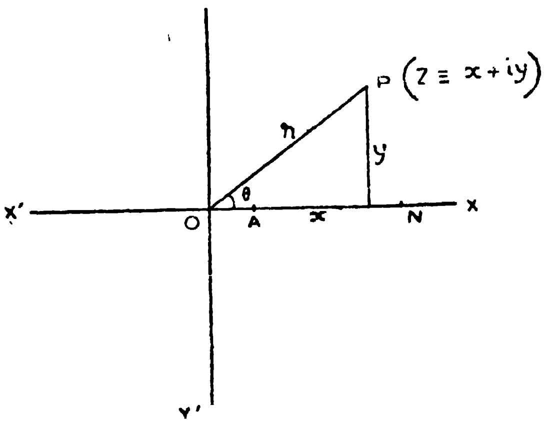
COMPLEX QUANTITIES

41. Complex Number and its Geometrical representation.

A quantity of the form $a+ib$, where a and b are real quantities, and $i^2=-1$, or, $i=\sqrt{-1}$ is called a *complex quantity*. If $b=0$, the quantity is purely real, whereas if $a=0$, the quantity is purely imaginary. A complex number is therefore the most general type of numbers which includes as special cases, purely real and purely imaginary numbers.

The complex quantities $a+ib$ and $a-ib$ are defined as *conjugate* to one another.

The complex number $a+ib$ can be zero only when a and b are both zeros.



In order to represent a real number geometrically, a straight line XOX' (called the real axis) is taken with

a fixed point O on it chosen as origin. A definite length OA measured along OX being taken to represent unit, a real number x is represented by a point N on OX such that $\frac{ON}{OA}=x$. If x be negative, N is to be taken to the left of O, and if positive, to the right. Thus, all real numbers will be represented by points on the straight line OX, *i.e.*, in one dimensional space.

For geometaical representation of complex numbers of the type $x+iy$, which involve two independent variables x and y, we naturally require a two-dimensional space. Let XOX' and YOY' be a set of rectangular axes in two-dimensional space, where XOX' may be taken as the real axis. To represent a purely imaginary number 'iy', we notice that if we take a line OM along OX to represent the real number y, then $i.iy$, or, $i^2y \equiv -y$ will be represented by an equal length along OX'. In other words, to represent i^2y we simply rotate OM (representing the real quantity y along OX) through two right angles. Multiplication by i twice successively amounting to a rotation through two right angles, it is naturally suggested that multiplication by i once would be best represented by a rotation through one right angle. Thus, to represent a purely imaginary quantity iy, the best way should be to measure off the real quantity y along OX, and then turn it through a right angle, so that 'iy' is represented by taking a length OM $(=y)$ along OY, which is called the imaginary axis.

A complex number $z \equiv x+iy$ being given, if a point P in the plane of the axes XOX', YOY' be taken with Cartesian co-ordinates x, y, then P corresponds uniquely to the number z and is called the point z which it geometrically represents. Thus, corresponding to every given complex quantity, there is a unique position of the point P which represents it, and conversely every point in the plane represents a definite complex number. In particular, points on the x-axis correspond to purely real numbers, for which y is zero, and points on y-axis to purely imaginary numbers, for which x is zero.

The figure containing the real and imaginary axes, in the plane of which complex numbers of the form 'z' are geometrically represented as above, is called the *Argand Diagram*, and the plane is spoken of as the z-plane.

42. Modulus and Amplitude.

Let r, θ be polar co-ordinates of P which represents a complex quantity $z \equiv x+iy$, so that r is the positive values of OP, and θ is the vertical angle XOP, traced out by a radius vector which revolves either positively or negatively from the initial position OX till it coincides with OP.

Then, r or $OP = +\sqrt{x^2+y^2}$ is called the *Modulus* of z and is written as mod. z or $|z|$; θ is called the *Amplitude or Argument* of z and is written as amp. z. The amplitude can evidently have an infinite number of values differing from each other by complete multiples of 2π. The value which satisfies the inequality

$$-\pi < \theta \leqslant \pi$$

is called the *Principal value of amp. z.*

Unless otherwise mentioned, by amplitude of a complex number we mean its principal value.

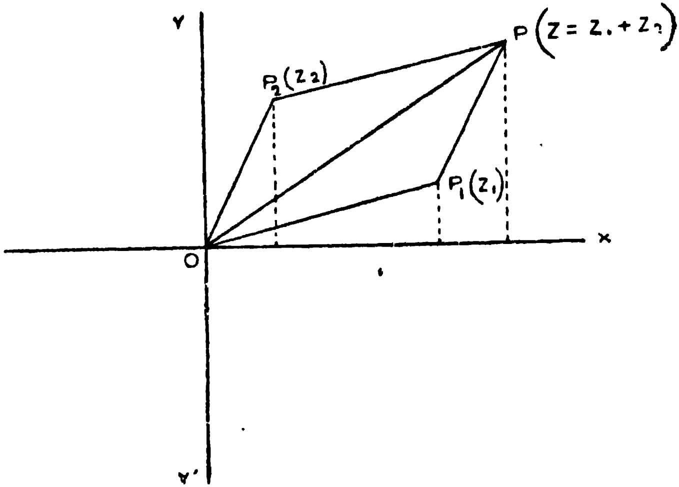
Since $x = r\cos\theta$, $y = r\sin\theta$, we evidently have

$$z = x+iy = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta} \qquad \text{[See } \textit{Art. 59} \text{]}$$

an equation which expresses the complex quantity z in terms of its modulus and amplitude.

43. Addition and subtraction of complex quantities.



Let P_1 and P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ respectively.

Complete the paralleogram OP_1PP_2 with OP_1 and OP_2 as adjacent sides. Then since the projection of OP on any line is equal to the algebraic sum of the projections of OP_1 and P_1P, *i.e.*. of OP_1 and OP_2, it is evident that the co-ordinates of P are $x_1 + x_2$ and $y_1 + y_2$ respectively. Hence, P represents the complex number $(x_1 + x_2) + i(y_1 + y_2)$.

But, $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$

$$= (x_1 + x_2) + i(y_1 + y_2).$$

Hence, the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides represents the sum of the two complex numbers represented by P_1 and P_2 in Argand's diagram.

A complex number $z = x + iy$ being represented by a point $P\ (x, y)$, the number $z = -x - iy$ will be represented by Q,

$(-x, -y)$ which is clearly the diametrically opposite point of P, obtained by producing PO through O to Q, making $OQ = OP$.

Now, subtraction of two complex numbers z_1 and z_2 may be treated as the addition of the numbers z_1 and $-z_2$ according to the parallelogram law as before. It can now be easily geometrically verified that

$$\text{if} \quad z = z_1 - z_2, \quad \text{then } z_1 = z + z_2,$$

$$\text{or,} \quad z_2 = z_1 - z.$$

With reference to addition and subtraction of complex quantities, two very important theorems are of great practical importance.

Theorem I. *The modulus of the sum of any number of complex quantities is less than or at most equal to the sum of their moduli.*

In other words,

$$|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. Since from geometry, $OP \ngtr^* OP_1 + P_1P$

$i.e.$, $\ngtr OP_1 + OP_2$, where P represents the sum of the complex numbers represented by P_1 and P_2, it follows that

$$|z_1 + z_2| \ngtr |z_1| + |z_2|.$$

In the same way,

$$|z_1 + z_2 + z_3| \leqslant |z_1 + z_2| + |z_3|$$

$$\leqslant |z_1| + |z_2| + |z_3|.$$

Similarly,

$$|z_1 + z_2 + \cdots\cdots + z_n| \leqslant |z_1| + |z_2| + \cdots\cdots + |z_n|.$$

*If OP_1 and OP_2 be in the same line, $OP = OP_1 + OP_2$.

Theorem II. *The modulus of the difference of two complex quantities is greater than or equal to the difference of their moduli,*

i.e., $| z_1 - z_2 | \qquad z_1 \qquad z_2$

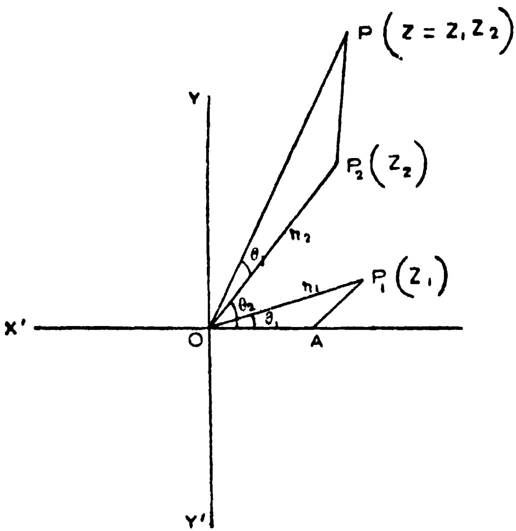
For, if $| z_1 | > | z_2 |$,

$$| z_1 | = | z_1 - z_2 + z_2 | \leqslant | z_1 - z_2 | + | z_2 |.$$

Hence, $| z_1 - z_2 | \geqslant | z_1 | - | z_2 |$.

Similarly, for the case $| z_2 | > | z_1 |$.

44. Multiplication of complex quantities.



Let z_1 and z_2 be two complex quantities, which expressed in terms of their moduli and amplitudes, are

$$z_1 \equiv r_1 (\cos \theta_1 + i \sin \theta_1), \; z_2 \equiv r_2 (\cos \theta_2 + i \sin \theta_2).$$

Now, the product of two complex quantities is defined in such a way that it follows the ordinary rules of multiplication in Algebra subject to the condition, $i^2 = -1$.

$$z \equiv z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

Similarly, if $z_1, z_2, z_3, \ldots, z_n$ be any number of complex quantities, their product will be a complex quantity given by

$$z \equiv z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= r_1 r_2 \ldots r_n [\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)]$$
$$= R(\cos\odot + i\sin\odot),$$

where $R = r_1 r_2 \ldots r_n$

$$\odot = \theta_1 + \theta_2 + \cdots + \theta_n.$$

Thus,

(*i*) *the modulus of the product of any number of complex quantities is equal to the product of their moduli ;*

(*ii*) *the amplitude of the product of any number of complex quantities is equal to the sum of their amplitudes (or differing from it by multiples of* 2π *if we consider the principal values in all cases).*

To express **geometrically** the operation of multiplying one complex number z_1 represented by the point P_1 by another z_2 represented by the point P_2, we see that if P represents the product z of z_1 and z_2, then its modulus

$$OP = OP_1 \,.\, OP_2$$

and $$\angle POX = \angle P_1OX + \angle P_2OX.$$

Now, if OA be taken along OX to represent unity, we get,

$$\frac{OP}{OP_2} = \frac{OP_1}{OA}, \text{ and } \angle POP_2 = \angle P_1OA.$$

Thus, the triangles POP_2, P_1OA are similar.

Hence, to get the product of the two complex quantities represented by P_1 and P_2, take OA along OX to represent unity and on OP_2 construct a triangle POP_2 similar to P_1OA. Then P represents the required product.

Another way is to turn OP_2 through an angle P_1OX, and then alter its length in the ratio $OP_1 : 1$, whereby OP will be obtained, representing the product.

Cor. A real quantity may be taken as a complex quantity of amplitude zero, of modulus equal to its magnitude. Hence, multiplication of a complex quantity z_1 by a real number k means simply a change in its modulus in the ratio $k : 1$, without altering its amplitude.

45. Division of one complex quantity by another.

The complex quantities $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$ being given, their quotient $z\left(\equiv \frac{z_1}{z_2}\right) = \rho(\cos\theta + i\sin\theta)$ is a complex quantity which when multiplied by z_2 gives z_1. Hence, from the above article

$$\rho r_2 = r_1 \text{ and } \theta_1 = \theta + \theta_2,$$

$$i.e., \quad \rho = \frac{r_1}{r_2}, \qquad \theta = \theta_1 - \theta_2$$

$$\text{and} \quad z \equiv \frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

Thus, *the modulus of the quotient of two complex quantities is the quotient of their moduli and the amplitude of the quotient is the difference of their amplitudes* (or differing by 2π).

Cor. As a particular case, the reciprocal of a complex quantity z is $\frac{1}{z}$ where $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and amp. $\left(\frac{1}{z}\right) = -$amp. z, as can be verified easily, remembering that the real number

$$1 = 1.(\cos 0 + i\sin 0).$$

46. De Moivre's Theorem. From Art. 44, it follows that if n be a positive integer,

$(\cos\theta + i\sin\theta)^n$ being the product of a complex number of modulus unity by itself n times in succession

$$= \cos(\theta+\theta+\cdots n \text{ times}) + i\sin(\theta+\theta+\cdots n \text{ times})$$

$$= \cos n\theta + i\sin n\theta,$$

which is De Moivre's theorem for a positive integral index [see *Art. 48*]. We can extend it to the case of any real index with necessary modification.

As multiplication by a complex quantity of the type $\cos\theta + i\sin\theta$ with modulus unity means a rotation through an angle θ [see *Art. 44* above], De Moivre's theorem for a positive integral index after all expresses algebraically the geometrical fact that to turn a line through the same angle θ, n times successively, has the same effect as turning the line through an angle $n\theta$.

47. We conclude with two very important theorems on the functions of complex numbers.

Theorem I. *Any rational function $R(z)$ of a complex variable $z(\equiv x+iy)$ can be reduced to the form $X+iY$ where X and Y are rational functions of x and y with real coefficients*

Proof. Any rational function $R(z)$ by definition of a rational function, is reducible to the form $\frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in z. A polynomial in z consists of terms containing integral powers of z with constant coefficients (which may be real or complex). Now, $z^n = (x+iy)^n$ for positive integral values of n can be reduced to the form $A+iB$, where A and B are rational functions of x and y, and this when multiplied by any coefficient, real or complex (of the form $a+ib$), retains the same form. Thus, $P(z)$ and $Q(z)$

ultimately reduce to the forms $A+iB$ and $C+iD$, where A, B, C, D are rational functions of the real quantities x and y.

Thus, $$R(z)=\frac{P(z)}{Q(z)}=\frac{A+iB}{C+iD}=\frac{(A+iB)(C-iD)}{C^2+D^2}$$

$$=\frac{AC+BD}{C^2+D^2}+i\,\frac{BC-AD}{C^2+D^2}.$$

Theorem II. *If $R(x+iy)=X+iY$, where R is a rational function of the complex quantity $x+iy$ with* **real** *coefficients, then $R(x-iy)=X-iY$.*

Proof. It is clear that $(x+iy)^n$, where n is a positive integer, is reducible to the form $A+iB$ where A and B are rational functions of the real quantities x and y, and replacing iy, by $-iy$, we easily see that $(x-iy)^n$ will reduce to $A-iB$. This continues to be true even if there be any real coefficients with $(x+iy)^n$. Thus, every term in any polynomial in $x+iy$ with *real* coefficients will reduce to the form $A+iB$, and the corresponding term of the same polynomial in $x-iy$ will reduce to $A-iB$.*

Hence, any polynomial in $x+iy$ with real coefficients being always reducible to the form A_1+iB_1, the same polynomial reduces to A_1-iB_1, when i is replaced by $-i$.

Now, $$R(x+iy)=\frac{P(x+iy)}{Q(x+iy)}$$ [where P and Q are polynomials which are in this case with real coefficients.]

$$=\frac{A+iB}{C+iD}=\frac{AC+BD}{C^2+D^2}+i\,\frac{BC-AD}{C^2+D^2}$$

$$\equiv X+iY$$ [*as in Theorem I*]

and $$R(x-iy)=\frac{P(x-iy)}{Q(x-iy)}=\frac{A-iB}{C-iD}$$ [from what has been stated above]

*This is not true if the coefficients are complex.

$$=\frac{(A-iB)(C+iD)}{C^2+D^2}=\frac{AC+BD}{C^2+D^2}-i\frac{BC-AD}{C^2+D^2}$$
$$=X-iY.$$

Hence the theorem.

EXAMPLES VI

1. Show that the representative points of the complex numbers $1+4i$, $2+7i$, $3+10i$ are collinear.

2. Prove that

$$(x+y+z)(x+y\omega^2+z\omega)(x+y\omega+z\omega^2)=x^3+y^3+z^3-3xyz,$$

where $\omega=\frac{1}{2}(-1+i\sqrt{3})$.

3. Express the following in the form $X+iY$, where X and Y are real numbers.

(i) $\left(\dfrac{1+i}{1-i}\right)^2$. (ii) $\left(\dfrac{a+ib}{a-ib}\right)^2-\left(\dfrac{a-ib}{a+ib}\right)^2$,

where a and b are real.

(iii) $\left(\dfrac{\alpha+\beta z}{\gamma+\delta z}\right)$,

where $z=x+iy$ and α, β, γ, δ, x, y are real quantities.

4. Prove that

$$|a+b|^2+|a-b|^2=2\{|a|^2+|b|^2\},$$

where a and b are any two complex quantities.

Deduce that

$$|a+\sqrt{a^2-b^2}|+|a-\sqrt{a^2-b^2}|=|a+b|+|a-b|.$$

5. If z_1 and z_2 are the roots of $\alpha z^2+2\beta z+\gamma=0$, then

$$|z_1|+|z_2|=\frac{1}{|\alpha|}\{|-\beta+\sqrt{\alpha\gamma}|+|-\beta-\sqrt{\alpha\gamma}|\}.$$

ANSWERS

3. (i) -1. (ii) $\dfrac{8iab(a^2-b^2)}{(a^2+b^2)^2}$. (iii) $\dfrac{(\alpha+\beta x)(\gamma+\delta x)+\beta\delta y^2+iy(\beta\gamma-\alpha\delta)}{(\gamma+\delta x)^2+\delta^2y^2}$.

CHAPTER VII

DE MOIVRE'S THEOREM

48. De Moivre's Theorem.*

For all real values of n, cos nθ + i sin nθ is a vaule of $(\cos\theta + i\sin\theta)^n$.

Written more fully, the theorem states that "*if n be integral, positive or negative, the value of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$; if n be a fraction, positive or negative, one of the values of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta+i\sin n\theta$* (the total number of values being equal to the denominator of n)".

Case I. When n is a **positive integer.**

By actual multiplication, we have

$$\begin{aligned}(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2) &= \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\\ &\quad + i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)\\ &=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2).\end{aligned}$$

Similarly,

$$\begin{aligned}(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3) &= \{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}(\cos\theta_3+i\sin\theta_3)\\ &=\cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3), \text{ as before.}\end{aligned}$$

Proceeding in this way, the product of the n factors

$$\begin{aligned}(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\cdots(\cos\theta_n+i\sin\theta_n)\\ =\cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots+\theta_n).\end{aligned}$$

*Called after the name of its discoverer *Abraham de Moivre* (1667-1754), who was of French descent but who settled in London where he gave lessons in Mathematics and ranked high as a Mathematician.

Put in this identity $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$.

Then, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Case II. When n is a **negative integer.**

Let $n = -m$, then m is a positive integer, and

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^m} \\ &= \frac{1}{\cos m\theta + i\sin m\theta}, \quad \text{[by } \textit{Case I} \text{]} \\ &= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} \\ &= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\ &= \cos m\theta - i\sin m\theta \\ &= \cos(-m)\theta + i\sin(-m)\theta \\ &= \cos n\theta + i\sin n\theta.\end{aligned}$$

Case III. When n is a **fraction**, positive or negative.

Let $n = \frac{p}{q}$, where q is a positive integer and p is any integer, positive or negative.

Now, $$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q\cdot\frac{\theta}{q}\right) + i\sin\left(q\frac{\theta}{q}\right),$$ [by *Case I*]

$$= \cos\theta + i\sin\theta.$$

$\therefore$ taking the qth root of both sides,

$\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is *one of the values of* $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$.

Raise each of these quantities to the pth power. Then, *one of the values of* $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$

$$=\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p,$$

i.e., $$=\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}. \quad [\text{ by } \textit{Case I and Case II}\,]$$

Hence, *one of the values of*

$$(\cos\theta + i\sin\theta)^n \text{ is } \cos n\theta + i\sin n\theta.$$

Thus, the theorem is completely established for *all rational values of n.*

Note. Even if n be *irrational*, $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$, the total number of different values in this case being infinite. The proof depends on the fact that the irrational number n can be defined in an indefinite number of ways as the limit of a convergent sequence of rational numbers. See Hobson's "Plane Trigonometry" and Hobson's "Theory of Functions of a Real Variable".

Cor. $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = (\cos\theta + i\sin\theta)^{-n}$,

$(\cos m\theta + i\sin m\theta)^n = (\cos\theta + i\sin\theta)^{mn}$.

Ex. 1. *Show that*

$$\frac{(\cos 2\theta + i\sin 2\theta)^3.(\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2.(\cos 4\theta + i\sin 4\theta)^{-3}} = 1.$$

We have,

$(\cos 2\theta + i\sin 2\theta)^3 = (\cos 6\theta + i\sin 6\theta) = (\cos\theta + i\sin\theta)^6$.

$(\cos 3\theta - i\sin 3\theta)^4 = \cos 12\theta - i\sin 12\theta = (\cos\theta + i\sin\theta)^{-12}$.

$(\cos 3\theta + i\sin 3\theta)^2 = \cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$.

$(\cos 4\theta + i\sin 4\theta)^{-3} = \cos 12\theta - i\sin 12\theta = (\cos\theta + i\sin\theta)^{-12}$.

$\therefore$ the given exp $= \dfrac{(\cos\theta + i\sin\theta)^6.(\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^6\,(\cos\theta + i\sin\theta)^{-12}} = 1.$

Ex. 2. *If* $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$,

prove that $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$$

It is known from Algebra that,

$$\text{if } a+b+c=0, \text{ then } a^3+a^3+c^3=3abc.$$

Let $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$ and $c=\cos\gamma+i\sin\gamma$, so that we have

$$a+b+c=\Sigma\cos\alpha+i\Sigma\sin\alpha=0, \text{ by the given condition.}$$

$$\therefore\quad (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$$
$$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma).$$

$\therefore$ by De Moivre's Theorem,

$$(\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$$
$$=3\cos(\alpha+\beta+\gamma)+3i\sin(\alpha+\beta+\gamma).$$

Hence, by equating the real and imaginary parts, the required results follow.

49. Extraction of any assigned root of a complex quantity by De Moivre's Theorem.

In the previous Article, it has been shown that $\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}$ is *one* of the values of $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$; the other values can be easily obtained. Since, the expression $\cos\theta+i\sin\theta$ remains unaltered if for θ we put $\theta+2r\pi$, where r is any integer, it follows that

$$(\cos\theta+i\sin\theta)^{\frac{1}{q}}=[\cos(2r\pi+\theta)+i\sin(2r\pi+\theta)]^{\frac{1}{q}}.$$

Since one of the values of the right-hand side quantity is

$$\cos\frac{2r\pi+\theta}{q}+i\sin\frac{2r\pi+\theta}{q},$$

$\therefore$ by giving to r in succession, the values 0, 1, 2, 3,... $(q-1)$, we see that each of the quantities

$$\cos\frac{\theta}{q}+i\sin\frac{\theta}{q},$$

$$\cos\frac{2\pi+\theta}{q}+i\sin\frac{2\pi+\theta}{q},$$

$$\cos\frac{4\pi+\theta}{q}+i\sin\frac{4\pi+\theta}{q}, \qquad \text{(A)}$$

$$\cos\frac{2(q-1)\pi+\theta}{q}+i\sin\frac{2(q-1)\pi+\theta}{q}$$

is equal to one of the values of $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$.

If values greater than $(q-1)$, *i.e.*, q, $q+1$, $q+2$,... be given to r, then we would get the same quantities already obtained in (A) repeated over and over again. Also no two of the quantities in (A) are the same, for no two of the angles involved therein can have the same sine and the same cosine, since, no two of these angles are equal, nor do they differ by a multiple of 2π.

Thus, by giving to r in succession, the values 0, 1, 2, 3,... $(q-1)$* in the expression

$$\cos\frac{2r\pi+\theta}{q}+i\sin\frac{2r\pi+\theta}{q}$$

q, *and only* q different values for $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$ are obtained.

This theorem may be usefully employed in extracting any assigned root of any quantity by putting it in De Moivre's form, that is, in the form $\cos\theta+i\sin\theta$.

*It may be easily seen that any q consecutive integral values may be given to r when the same set of values occurring in a different order will be obtained for the expression.

50. To put x+iy in De Moivre's form.

Suppose $x+iy=r(\cos\theta+i\sin\theta)$;

then, equating real and imaginary parts, we have

$$r\cos\theta=x, \qquad \cdots \qquad \cdots \quad (1)$$

$$r\sin\theta=y. \qquad \cdots \qquad \cdots \quad (2)$$

Squaring (1) and (2) and adding, $r^2=x^2+y^2$.

$$\therefore \quad r=\sqrt{x^2+y^2}. \qquad \cdots \qquad \cdots \quad (3)$$

From (1) and (2), $\cos\theta=\dfrac{x}{r}=\dfrac{x}{\sqrt{x^2+y^2}}$ $\cdots$ (4)

$$\sin\theta=\frac{y}{r}=\frac{y}{\sqrt{x^2+y^2}} \cdots \quad (5)$$

Thus, r and θ are determined.

Particular cases.

(1) $1=\cos 0+i\sin 0$

(2) $-1=\cos\pi+i\sin\pi$

(3) $i=\cos\frac{1}{2}\pi+i\sin\frac{1}{2}\pi$

(4) $-i=\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi$

or, $\quad =\cos\frac{1}{2}\pi-i\sin\frac{1}{2}\pi.$

Note 1. It is convenient to take the positive value of the square root $\sqrt{x^2+y^2}$; then θ must be taken in that quadrant which makes $\cos\theta$ of the same sign as x and $\sin\theta$ of the same sign as of y. Whatever be the values of x and y, there is one and only one value of θ, lying between $-\pi$ radians and $+\pi$ radians, which satisfies the equations (4) and (5).

Note 2. Since the expression $\cos\theta+i\sin\theta$ remains unaltered if for θ we put $\theta+2n\pi$, where n is an integer, $x+iy$ can also be expressed in the more *general form* $r\{\cos(2n\pi+\theta)+i\sin(2n\pi+\theta)\}$.

Ex. 3. *Find all the values of* (i) $(1+i)^{\frac{1}{3}}$. [*C. P. 1932, '38, '44.*]

(ii) $1^{\frac{1}{3}}$.

(i) Suppose $1+i=r(\cos\theta+i\sin\theta)$; then $r\cos\theta=1$, $r\sin\theta=1$.

$\therefore \quad r^2=2$ and $r=\sqrt{2}$; $\cos\theta=\dfrac{1}{\sqrt{2}}$, $\sin\theta=\dfrac{1}{\sqrt{2}}$. $\therefore\ \theta=\frac{1}{4}\pi$.

$\therefore \quad 1+i=\sqrt{2}\,(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi)$,

and $\quad (1+i)^{\frac{1}{3}}=2^{\frac{1}{6}}\,(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi)^{\frac{1}{3}}$

$$=2^{\frac{1}{6}}\,\{\cos(2n\pi+\tfrac{1}{4}\pi)+i\sin(2n\pi+\tfrac{1}{4}\pi)\}^{\frac{1}{3}}$$

$$=2^{\frac{1}{6}}\left\{\cos\frac{(8n+1)\pi}{12}+i\sin\frac{(8n+1)\pi}{12}\right\}.$$

Putting $n=0, 1, 2$, the required values are obtained.

(ii) $1^{\frac{1}{3}}=(\cos 0+i\sin 0)^{\frac{1}{3}}=(\cos 2n\pi+i\sin 2n\pi)^{\frac{1}{3}}$

$$=\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}.$$

Giving to n the values, 0, 1, 2, the required values are

$$\cos 0+i\sin 0,\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3},$$

i.e., $\quad 1,\ \frac{1}{2}(-1+\sqrt{-3}),\ \frac{1}{2}(-1-\sqrt{-3})$.

Ex. 4. *Prove that the roots of the equation* $x^{10}+11x^5-1=0$ *are the values of* $\pm\dfrac{\sqrt{5}-1}{2}\left(\cos\dfrac{2r\pi}{5}\pm i\sin\dfrac{2r\pi}{5}\right)$ *where r is an integer.*

[*C. H. 1932.*]

From the given equation, we get

$$x^5=\frac{-11\pm\sqrt{125}}{2}=\frac{-11\pm5\sqrt{5}}{2}=\frac{-176\pm80\sqrt{5}}{32}$$

$$=\left(\pm\frac{\sqrt{5}-1}{2}\right)^5=\left(\pm\frac{\sqrt{5}-1}{2}\right)^5(\cos 2r\pi\pm i\sin 2r\pi).$$

$$\therefore \quad x=\pm\frac{\sqrt{5}-1}{2}\{\cos 2r\pi\pm i\sin 2r\pi\}^{\frac{1}{5}}$$

which by De Moivre's Theorem

$$=\pm\frac{\sqrt{5}-1}{2}\left(\cos\frac{2r\pi}{5}\pm i\sin\frac{2r\pi}{5}\right),$$

where r is any integer.

Note. It may be seen that for different integral values of r, positive or negative, the expression has only ten different values which are the ten roots of the given equation.

Ex. 5. *If* $x + \frac{1}{x} = 2\cos\theta$, *show that* $x^n + \frac{1}{x^n} = 2\cos n\theta$.

$$x + \frac{1}{x} = 2\cos\theta.$$

$$\therefore\quad x^2 - 2x\cos\theta + 1 = 0,$$

whence, $x = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta$.

Taking the $+ve$ sign, $x^n + \frac{1}{x^n} = x^n + x^{-n}$

$$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$

$$= \{\cos n\theta + i\sin n\theta\} + \{\cos(-n\theta) + i\sin(-n\theta)\}$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta.$$

Similarly, with $-ve$ sign, the same result easily follows.

Ex. 6. *If* $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, *then*

$$\Sigma\cos 4\alpha = 2\Sigma\cos 2(\beta+\gamma),\ \Sigma\sin 4\alpha = 2\Sigma\sin 2(\beta+\gamma).$$

Let $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$.

$$\therefore\quad x + y + z = \Sigma\cos\alpha + i\,\Sigma\sin\alpha = 0.$$

$$\therefore\quad 2y^2z^2 + 2z^2x^2 + 2x^2y^2 - x^4 - y^4 - z^4 = 0,$$

i.e., $\Sigma x^4 = 2\,\Sigma\, y^2z^2$, or, $\Sigma(\cos\alpha + i\sin\alpha)^4$

$$= 2\,\Sigma(\cos\beta + i\sin\beta)^2(\cos\gamma + i\sin\gamma)^2,$$

or, by De Moivre's Theorem,

$$\Sigma(\cos 4\alpha + i\sin 4\alpha) = 2\,\Sigma(\cos 2\beta + i\sin 2\beta)(\cos$$

$$= 2\,\Sigma\{\cos 2(\beta+\gamma) + i\sin 2(\beta+\gamma)\}.$$

Now equating real and imaginary parts on both sides, the results follow.

EXAMPLES VII

1. (i) Find the general value of θ which satisfies the equation

$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\cdots(\cos n\theta + i\sin n\theta) = 1.$$

(ii) Express $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^5$ in the form $x + iy$.

2. (i) If $x + \dfrac{1}{x} = 2\cos\theta$, and $y + \dfrac{1}{y} = 2\cos\phi$, prove that $\cos(\theta+\phi)$ is one of the values of $\dfrac{1}{2}\left(xy + \dfrac{1}{xy}\right)$ and that one of the values of $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2\cos(m\theta - n\phi)$.

(ii) If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$, find the values of $\cos(\theta+\phi)$ and $\cos(\theta-\phi)$ in terms of a, b.

3. Prove that

(i) $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = 2(a^2+b^2)^{\frac{m}{2n}}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right).$

(ii) $$\left\{\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right\}^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$$
$$= (\sin\theta + i\cos\theta)^n.$$

[*C. P. 1946.*]

4. If $x_r = \cos\dfrac{\pi}{2^r} + i\sin\dfrac{\pi}{2^r}$, prove that

$x_1 x_2 x_3 \ldots$ to infinity $= -1$. [*C. P. 1943, '51.*]

5. If $(a_1 + ib_1)(a_2 + ib_2)\cdots(a_n + ib_n) = A + iB$,

(i) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + \cdots + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}.$

(ii) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\cdots(a_n^2 + b_n^2) = A^2 + B^2.$

6. Find the equation whose roots are the nth powers of the roots of the equation

$$x^2 - 2x \cos\theta + 1 = 0.$$

7. If $2\cos\alpha = a + a^{-1}$, $2\cos\beta = b + b^{-1}$ etc., show that under certain conditions,

(i) $2\cos(\alpha+\beta+\gamma+\cdots) = abc\cdots + \dfrac{1}{abc\cdots}$. [*C. P. 1934.*]

(ii) $2\cos(p\alpha + q\beta + r\gamma + \cdots) = a^p b^q c^r \cdots + \dfrac{1}{a^p b^q c^r \cdots}$.

8. Simplify

$$\{\cos\alpha - \cos\beta + i(\sin\alpha - \sin\beta)\}^n + \{\cos\alpha - \cos\beta - i(\sin\alpha - \sin\beta)\}^n.$$

9. If $(1+x)^n = p_0 + p_1 x + p_2 x^2 + \cdots$, show that

$$p_0 - p_2 + p_4 - \cdots = 2^{\frac{n}{2}} \cos \tfrac{1}{4} n\pi.$$

$$p_1 - p_3 + p_5 - \cdots = 2^{\frac{n}{2}} \sin \tfrac{1}{4} n\pi.$$

[*C. H. 1937.*]

10. Find all the values of

(i) $(1+i)^{\frac{1}{7}}$. [*C. P. 1931.*]

(ii) $(1+i)^{\frac{1}{5}}$. [*C. P. 1940.*]

(iii) $(-i)^{\frac{1}{6}}$.

(iv) $(-1)^{\frac{2}{5}}$.

11. Prove that the nth roots of unity [illegible] in G. P.

12. Using De Moivre's Theorem, solve the equations

(i) $x^7 + x^4 + x^3 + 1 = 0$.

(ii) $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

13. Prove that $\sqrt[n]{a+ib}+\sqrt[n]{a-ib}$ has n real values, and find those of $\sqrt[3]{1+\sqrt{-3}}+\sqrt[3]{1-\sqrt{-3}}$.

14. If n be a positive integer, prove that

$$(1+i)^n+(1-i)^n=2^{\frac{n}{2}+1}\cos\tfrac{1}{4}n\pi.$$

15. (i) If $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, and $z=\cos\gamma+i\sin\gamma$, and if $x+y+z=0$, then $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$.

(ii) If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, then $\Sigma\sin^2\alpha=\Sigma\cos^2\alpha=\frac{3}{2}$.

16. (i) Solve $x^7=1$ by the help of De Moivre's Theorem. [*C. P. 1939, '43.*]

(ii) Prove that the sum of the nth powers of the roots of the above equation, n being an integer not divisible by 7, is zero.

17. If $x=\cos\theta+i\sin\theta$ and $1+\sqrt{1-a^2}=na$, prove that

$$1+a\cos\theta=\frac{a}{2n}\left(1+nx\right)\left(1+\frac{n}{x}\right).$$

18. If $a=\cos\dfrac{2r\pi}{n}+i\sin\dfrac{2r\pi}{n}$, and if r and p are prime to n, prove that

$$1+a^p+a^{2p}+\cdots\cdots+a^{(n-1)p}=0. \qquad [\textit{ C. H. 1938.}]$$

19. If $(1+x)^n=a_0+a_1x+a_2x^2+\cdots\cdots$, ($n$ being a positive integer), prove that

$$a_0+a_4+a_8+\cdots=2^{n-2}+2^{\frac{1}{2}n-1}\cos\tfrac{1}{4}n\pi.$$

20. If $a=\cos\theta_1+i\sin\theta_1$, $b=\cos\theta_2+i\sin\theta_2$,

$c=\cos\theta_3+i\sin\theta_3$ and $a+b+c=abc$, then

$$\cos(\theta_2-\theta_3)+\cos(\theta_3-\theta_1)+\cos(\theta_1-\theta_2)+1=0.$$

21. If $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, and $z=\cos\gamma+i\sin\gamma$, show that $\dfrac{(y+z)(z+x)(x+y)}{xyz}$ is real, and is equal to $8\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)\cos\frac{1}{2}(\alpha-\beta)$.

22. By using substitutions of the form $x=\cos 2\theta+i\sin 2\theta$ in the identity
$(x_1-x_2)(x_3-x_4)+(x_2-x_3)(x_1-x_4)+(x_3-x_1)(x_2-x_4)=0$,
show that

$$\sin(\theta_1-\theta_2)\sin(\theta_3-\theta_4)+\cdots+\cdots=0.$$

23. From the identity

$$\frac{x^{n+1}-y^{n+1}}{x-y}=x^n+x^{n-1}y+x^{n-2}y^2+\cdots+y^n$$

deduce the sum of all the values of $\cos(p\alpha+q\beta)$, where p and q are positive integers, such that $p+q=n$.

[*Put $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$ and equate real parts of both sides.*]

ANSWERS

1. (i) $\dfrac{4m\pi}{n(n+1)}$. (ii) $\sin 10\theta-i\cos 10\theta$. **2.** (ii) $\dfrac{1}{2}\left(ab+\dfrac{1}{ab}\right)$; $\dfrac{1}{2}\dfrac{a^2+b^2}{ab}$.

6. $x^2-2x\cos n\theta+1=0$. **8.** $2^{n+1}\sin^n\dfrac{\alpha-\beta}{2}\cos\dfrac{n}{2}(\pi+\alpha+\beta)$.

10. (i) $2^{\frac{1}{14}}\left\{\cos\dfrac{1}{7}\left(2n\pi+\dfrac{\pi}{4}\right)+i\sin\dfrac{1}{7}\left(2n\pi+\dfrac{\pi}{4}\right)\right\}$, $n=0, 1, 2, 3, 4, 5, 6$.

(ii) $2^{\frac{1}{10}}\left\{\cos\dfrac{1}{5}\left(2n\pi+\dfrac{\pi}{4}\right)+i\sin\dfrac{1}{5}\left(2n\pi+\dfrac{\pi}{4}\right)\right\}$, $n=0, 1, 2, 3, 4$.

(iii) $\cos(4n+3)\dfrac{\pi}{12}+i\sin(4n+3)\dfrac{\pi}{12}$, $n=0, 1, 2, 3, 4, 5$.

(iv) $\cos\dfrac{2r\pi}{5}+i\sin\dfrac{2r\pi}{5}$, $r=0, 1, 2, 3, 4$.

12. (i) $-1, \cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3}, \ \pm\left(\cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}\right)$.

(ii) $\cos\frac{2r\pi}{7} + i \sin\frac{2r\pi}{7}, \ r = 1, 2, 3, 4, 5, 6.$

18. $2^{\frac{4}{3}} \cos\frac{(6n+1)\pi}{9}, \ n = 0, 1, 2.$

16. (i) $\cos\frac{2n\pi}{7} + i \sin\frac{2n\pi}{7}, \ (n = 0, 1, 2, 3, 4, 5, 6).$

23. $\dfrac{\sin\left\{\left(n+1\right)\frac{\alpha-\beta}{2}\right\}}{\sin\frac{\alpha-\beta}{2}} \cos\left(n\frac{\alpha+\beta}{2}\right).$

CHAPTER VIII

IMPORTANT DEDUCTIONS FROM DE MOIVRE'S THEOREM

51. Expansions of sin nθ and cos nθ *(n being a positive integer).*

When n is a positive integer, we have

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

Since n is a positive integer, therefore, expanding the right side by the Binomial theorem,* we have

$\cos n\theta + i \sin n\theta$

$$= \cos^n\theta + n \cos^{n-1}\theta . i \sin \theta + \frac{n(n-1)}{2\,!} \cos^{n-2}\theta . i^2 \sin^2\theta$$
$$+ \frac{n(n-1)(n-2)}{3\,!} \cos^{n-3}\theta . i^3 \sin^3\theta + \cdots\cdots$$

Since, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \ldots$

$\therefore \quad \cos n\theta + i \sin n\theta$

$$= \cos^n\theta - \frac{n(n-1)}{2\,!} \cos^{n-2}\theta \sin^2\theta + \cdots\cdots$$
$$+ i\left[n \cos^{n-1}\theta \sin \theta - \frac{n(n-1)(n-2)}{3\,!} \cos^{n-3}\theta \sin^3\theta + \cdots\right].$$

Equating real and imaginary parts, we have,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2\,!} \cos^{n-2}\theta \sin^2\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4\,!} \cos^{n-4}\theta \sin^4\theta - \cdots \qquad (1)$$

*See Art. 68.

$$\text{and } \sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3\,!}\cos^{n-3}\theta\sin^3\theta$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{5\,!}\cos^{n-5}\theta\sin^5\theta - \cdots \quad (2)$$

Obs. The terms in each of the above two series are alternately positive and negative and each series continues till one of the factors in the numerator is zero.

The last term in the expansions of $\sin n\theta$ and $\cos n\theta$ are respectively

$$(-1)^{\frac{n-1}{2}}\sin^n\theta \text{ and } (-1)^{\frac{n-1}{2}} n\cos\theta\sin^{n-1}\theta, \text{ when } n \text{ is } odd\,;$$

$$(-1)^{\frac{n-2}{2}} n\cos\theta\sin^{n-1}\theta \text{ and } (-1)^{\frac{n}{2}}\sin^n\theta, \text{ when } n \text{ is } even.$$

Note. The values of $\cos n\theta$ and $\sin n\theta$ can also be obtained by the method of Induction without the use of imaginary quantities.

52. Expansion of tan $n\theta$.

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}. \quad \cdots \quad \cdots \quad (1)$$

Writing down the expansions of $\sin n\theta$ and $\cos n\theta$ in (1) and then dividing the numerator and the denominator by $\cos^n\theta$, we get,

$$\tan n\theta = \frac{{}^nc_1\tan\theta - {}^nc_3\tan^3\theta + \cdots}{1 - {}^nc_2\tan^2\theta + {}^nc_4\tan^4\theta - \cdots}$$

Note. When n is *odd*, the last term in the numerator is $(-1)^{\frac{n-1}{2}}\tan^n\theta$ and that in the denominator is $(-1)^{\frac{n-1}{2}} n\tan^{n-1}\theta$, when n is *even*, the last term in the numerator is $(-1)^{\frac{n-2}{2}} n\tan^{n-1}\theta$, and that in the denominator is $(-1)^{\frac{n}{2}}\tan^n\theta$.

53. Expansion of cos α *(in ascending powers of α).*

By De Moivre's Theorem, when n is a positive integer,

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Expanding the right-hand side by the Binomial Theorem and equating real parts on both sides, we get, as in Art. 51,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2\,!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4\,!}\cos^{n-4}\theta\sin^4\theta - \cdots$$

Put $n\theta = \alpha$, so that $n = \frac{\alpha}{\theta}$; then we have

$$\cos\alpha = \cos^n\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)}{2\,!}\cos^{n-2}\theta\sin^2\theta + \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)}{4\,!}\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{2\,!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{4\,!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \cdots(1)$$

Now keeping α constant, make n an infinitely large positive integer; θ accordingly diminishes indefinitely and $\frac{\sin\theta}{\theta}$ and $\cos\theta$ as also every power of these become in the limit equal to 1.*

$\therefore$ from (1), we have,

$$\mathbf{\cos\alpha = 1 - \frac{\alpha^2}{2\,!} + \frac{\alpha^4}{4\,!} - \frac{\alpha^6}{6\,!} + \cdots\cdots + (-1)^n\frac{\alpha^{2n}}{(2n)\,!} + \cdots \text{ ad inf.}}$$

*See Chapter XV, Section E. Also see Appendix, for a discussion on the method of proof and for Convergence test.

54. Expansion of sin α.

By De Moivre's Theorem, when n is a positive integer,

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

Expanding the right-hand side by the Binomial Theorem and equating imaginary parts on both sides, we get, as in Art. 51,

$$\sin n\theta = n \cos^{n-1}\theta \sin \theta - \frac{n(n-1)(n-2)}{3\,!} \cos^{n-3}\theta \sin^3\theta + \cdots$$

Putting $n\theta = \alpha$ and simplifying as before, we have

$$\sin \alpha = \alpha \cos^{n-1}\theta \left(\frac{\sin \theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{3\,!} \cos^{n-3}\theta \left(\frac{\sin \theta}{\theta}\right)^3 + \cdots$$

Now, making n an indefinitely large positive integer and keeping α constant, we have, as before,

$$\sin \alpha = \alpha - \frac{\alpha^3}{3\,!} + \frac{\alpha^5}{5\,!} - \frac{\alpha^7}{7\,!} + \cdots\cdots + (-1)^n \frac{\alpha^{2n+1}}{(2n+1)\,!} + \cdots \text{ ad inf.}$$

55. In establishing the series for sin α and cos α, it is tacitly assumed that α is the *radian measure* of the angle considered; for it is only when the angle θ is expressed in radian measure, that $\dfrac{\sin \theta}{\theta} = 1$, θ being indefinitely diminished. However, it is easy to obtain the requisite modification of the above formulæ when the angle is expressed in any other system of measurement.

Let us obtain *the expansion of sin α°.*

Now, $\quad \alpha^\circ = \alpha \dfrac{\pi}{180}$ radians.

$$\therefore \quad \sin a^\circ = \sin \frac{a\pi}{180}$$

$$= \frac{a\pi}{180} - \frac{1}{3\,!}\left(\frac{a\pi}{180}\right)^3 + \frac{1}{5\,!}\left(\frac{a\pi}{180}\right)^5 - \cdots$$

Similarly, $\cos a^\circ = 1 - \dfrac{1}{2\,!}\left(\dfrac{a\pi}{180}\right)^2 + \dfrac{1}{4\,!}\left(\dfrac{a\pi}{180}\right)^4 - \cdots$

56. Addition Formulæ for any number of angles.

By the use of De Moivre's Theorem, general formulæ for the sine, cosine, tangent of the sum of any number of unequal angles can be easily deduced.

$$(\cos a_1 + i \sin a_1)(\cos a_2 + i \sin a_2)\cdots(\cos a_n + i \sin a_n)$$
$$= \cos(a_1 + a_2 + \cdots + a_n) + i \sin(a_1 + a_2 + \cdots + a_n) \quad \cdots \ (1)$$

Now, $\cos a_1 + i \sin a_1 = \cos a_1 \,(1 + i \tan a_1)$

$\cos a_2 + i \sin a_2 = \cos a_2 \,(1 + i \tan a_2)$,

and so on.

Hence, (1) may be written as

$$\cos(a_1 + a_2 + \cdots + a_n) + i \sin(a_1 + a_2 + \cdots + a_n)$$
$$= \cos a_1 \cos a_2 \ldots \cos a_n \,(1 + i \tan a_1)(1 + i \tan a_2)\ldots$$
$$(1 + i \tan a_n) \ldots \ (2)$$
$$= \cos a_1 \cos a_2 \ldots \cos a_n [1 + is_1 + i^2 s_2 + i^3 s_3 + i^4 s_4 + i^5 s_5 + \cdots]$$
$$= \cos a_1 \cos a_2 \ldots \cos a_n \,[1 + is_1 - s_2 - is_3 + s_4 + is_5 + \cdots],$$

where s_r denotes the sum of the products of $\tan a_1$, $\tan a_2, \ldots \tan a_n$, taken r at a time.

Now, equating real and imaginary parts, we get

$$\cos(a_1 + a_2 + \cdots + a_n)$$
$$= \cos a_1 \cos a_2 \ldots \cos a_n \,(1 - s_2 + s_4 - \cdots) \quad \cdots \ (3)$$

$$\sin(a_1 + a_2 + \cdots + a_n)$$
$$= \cos a_1 \cos a_2 \ldots \cos a_n \,(s_1 - s_3 + s_5 - \cdots) \quad \cdots \ (4)$$

and by division,

$$\tan(a_1 + a_2 \cdots + a_n) = \frac{s_1 - s_3 + s_5 - s_7 + \cdots}{1 - s_2 + s_4 - s_6 + \cdots} \quad \cdots \ (5)$$

Note 1. If n be *odd*, the last term in the numerator is $(-1)^{\frac{n-1}{2}} s_n$ and that in the denominator is $(-1)^{\frac{n-1}{2}} s_{n-1}$; and if n be *even*, the last term in the numerator is $(-1)^{\frac{n-2}{2}} s_{n-1}$; and that in the denominator is $(-1)^{\frac{n}{2}} s_n$.

Note 2. By putting $\alpha_1 = \alpha_2 = \alpha_3 \cdots = \alpha_n = \theta$, in (3), (4), (5) the formulæ for $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, can be easily deduced.

Ex. 1. *If* $\dfrac{\sin\theta}{\theta} = \dfrac{5045}{5046}$, *show that* $\theta = 1°58'$ *nearly.* [*C. P. 1926.*]

When θ is very small, $\dfrac{\sin\theta}{\theta}$ is very nearly equal to 1; here $\dfrac{\sin\theta}{\theta}$ is very nearly equal to 1; hence, θ must be small. Therefore, in the series for $\sin\theta$, neglecting the powers of θ higher than the 3rd, we have,

$$\frac{\sin\theta}{\theta} = \frac{1}{\theta}\left(\theta - \frac{\theta^3}{3!}\right) = 1 - \frac{\theta^2}{6}.$$

$$\therefore \quad 1 - \frac{\theta^2}{6} = \frac{5045}{5046} \text{ and hence } \frac{\theta^2}{6} = 1 - \frac{5045}{5046} = \frac{1}{5046}.$$

Thus, $\theta^2 = \dfrac{1}{841}$. $\therefore\ \theta = \dfrac{1}{29}$ radian $= 1°\ 58'$ nearly.

Ex. 2. *Expand* $\tan x$ *in powers of* x *as far as the term involving* x^5.

$$\tan x = \frac{\sin x}{\cos x} = \sin x\,(\cos x)^{-1}$$

$$= \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right\}\left\{1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)\right\}^{-1}.$$

Now, expanding by Binomial theorem and simplifying,

we get $\quad \mathbf{\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\cdots}$

Ex. 3. *Expand* $\cos^2\theta$ *in powers of* θ.

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2\theta)^2}{2!} + \frac{(2\theta)^4}{4!} - \cdots\right]$$

$$= \frac{1}{2}\left[2 - \frac{2^2\theta^2}{2!} + \frac{2^4\theta^4}{4!} - \cdots\right] = 1 - \frac{2\theta^2}{2!} + \frac{2^3\theta^4}{4!} - \cdots$$

Ex. 4. *Prove that*

$$\sin^2\theta \cos\theta = \theta^2 - \tfrac{5}{6}\theta^4 + \cdots + (-1)^{n+1}\frac{3^{2n}-1}{4(2n)\,!}\,\theta^{2n} + \cdots$$

[*C. P. 1935.*]

$$4\sin^2\theta\cos\theta = 2\sin\theta .\ 2\sin\theta\cos\theta = 2\sin\theta .\ \sin 2\theta$$

$$= \cos\theta - \cos 3\theta$$

$$= 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \cdots\cdots + (-1)^n \frac{\theta^{2n}}{(2n)\,!} + \cdots\cdots$$

$$- \left\{1 - \frac{3^2\theta^2}{2\,!} + \frac{3^4\theta^4}{4\,!} - \cdots\cdots + (-1)^n \frac{3^{2n}\theta^{2n}}{(2n)\,!} + \cdots\right\}$$

$$= 4\theta^2 - \frac{20\theta^4}{6} + \cdots\cdots + (-1)^{n+1}\frac{3^{2n}-1}{(2n)\,!}\,\theta^{2n} - \cdots$$

$$\therefore\ \sin^2\theta\cos\theta = \theta - \tfrac{5}{6}\theta^4 + \cdots + (-1)^{n+1}\frac{3^{2n}-1}{4(2n)\,!}\,\theta^{2n} + \cdots$$

Ex. 5. *Find the limiting value of* $\dfrac{\tan 2\theta - 2\sin\theta}{\theta^3}$ *when* θ *tends to zero.*

Writing down the series for $\sin\theta$ and $\tan 2\theta$ (see *Ex. 2 above*) the given expression becomes

$$\frac{\left\{2\theta + \frac{1}{3}(2\theta)^3 + \frac{2}{15}(2\theta)^5 + \cdots\right\} - 2\left\{\theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \cdots\right\}}{\theta^3}$$

$$= \frac{3\theta^3 + \text{terms involving } \theta^5 \text{ and higher powers of } \theta}{\theta^3}$$

$$= 3 + \text{terms involving } \theta^2 \text{ and higher powers of } \theta.$$

Hence, the required limit is 3.

Ex. 6. *If* $m = u - e\sin u$, *and if* e *be so small that its powers above the second may be neglected, show that*

$$u = m + e\sin m + \tfrac{1}{2}e^2\sin 2m.$$

Here, $u = m + e\sin u$, $\qquad\cdots\qquad\cdots\quad$ (i)

and since e is small, u differs from m by a small quantity of the order of e.

Let $u=m+x$, so that x is of the order of e.

Then, from (i),

$$m+x=m+e\sin(m+x)$$

$$=m+e\sin m\cos x+e\cos m\sin x$$

$$=m+e\sin m\left\{1-\frac{x^2}{2!}+\cdots\right\}+e\cos m\left\{x-\frac{x^3}{3!}+\cdots\right\}$$

$$=m+e\sin m+x\,e\cos m \qquad \text{[neglecting terms of an order higher than } e^2.\text{]}$$

whence

$$x=\frac{e\sin m}{1-e\cos m}=e\sin m\,(1-e\cos m)^{-1}$$

$$=e\sin m\,(1+e\cos m+\cdots) \qquad \text{[by Binomial Theorem]}$$

$$=e\sin m+e^2\sin m\cos m \quad \text{[neglecting higher powers of } e\text{]}$$

$$=e\sin m+\tfrac{1}{2}e^2\sin 2m.$$

$$\therefore \quad u=m+x=m+e\sin m+\tfrac{1}{2}e^2\sin 2m.$$

Note. The above process is known as the *Reversal of series.*

EXAMPLES VIII

1. (i) Expand $\sin^3 x$ and $\cos^4 x$ in powers of x.

[*Use* $\sin^3 x=\frac{1}{4}(3\sin x-\sin 3x)$; $\cos^4 x=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$.]

(ii) Expand $\sin^2 x$ and $\cos^3 x$ in powers of x.

[*Use* $\sin^2 x=\frac{1}{2}(1-\cos 2x)$; $\cos^3 x=\frac{1}{4}(3\cos x+\cos 3x)$.]

2. Express $\sin\left(\dfrac{\pi}{4}+x\right)\cos x$ as a power series in x.

3. Use the fact that $\cos 2\theta+i\sin 2\theta=(\cos\theta+i\sin\theta)^2$ to find the formulæ for $\cos 2\theta$ and $\sin 2\theta$.

By a similar method find formulæ for $\cos 3\theta$ and $\sin 3\theta$.

4. (i) If $\dfrac{\sin\theta}{\theta}=\dfrac{2165}{2166}$, show that θ is nearly $3°$.

(ii) Calculate the value of $\sin 3°$ correct to 3 significant figures.

5. (i) If $\sin x = \cdot 9998$, show that $x = 88° 51'$ nearly.

(ii) If $\cos(\frac{1}{3}\pi + x) = \cdot 49$, show that $x = 39'7'$ nearly.

(iii) If $\cos x = x$, show that $x = \cdot 7$ nearly.

(iv) If $\tan\theta = \frac{1}{15}$, find an approximate value for θ.

6. (i) Express $\sin 5x$ in powers of $\sin x$.

(ii) Express $\cos 5\theta$ in powers of $\cos\theta$. [*C. P. 1931.*]

(iii) Prove that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$$

(iv) Prove that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

7. (i) Prove that if $-\frac{1}{4}\pi < \theta < \frac{1}{4}\pi$, then

$$\sin n\theta = \cos^n\theta\{{}^nc_1\tan\theta - {}^nc_3\tan^3\theta + \cdots\}$$
$$\cos n\theta = \cos^n\theta\{1 - {}^nc_2\tan^2\theta + {}^nc_4\tan^4\theta - \cdots\}.$$

[*$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \{\cos\theta\,(1 + i\tan\theta)\}^n$; expand right side and equate real and imaginary parts.*]

(ii) Show that (n being a positive integer)

$$1 - {}^nc_2 + {}^nc_4 - \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

and $${}^nc_1 - {}^nc_3 + {}^nc_5 - \cdots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$ [*C. P. 1941.*]

8. Use De Moivre's Theorem to prove that $\cos\frac{1}{7}\pi$, $\cos\frac{3}{7}\pi$ and $\cos\frac{5}{7}\pi$ are roots of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Hence, prove that

$$\cos\tfrac{1}{7}\pi + \cos\tfrac{3}{7}\pi + \cos\tfrac{5}{7}\pi = \tfrac{1}{2}.$$

9. Prove that $\theta\cot\theta = 1 - \frac{1}{3}\theta^2 - \frac{1}{45}\theta^4$, if θ is small.

10. Find the limiting values of the following expressions when x tends to zero.

(i) $\dfrac{x - \sin x}{x^3}$. (ii) $\dfrac{\tan x - \sin x}{\sin^3 x}$.

(iii) $\dfrac{\tan 2x - 2\tan x}{2x}$. [*C. P. 1933.*]

(iv) $\dfrac{3\sin x - \sin 3x}{x - \sin x}$ [*C. P. 1936.*]

11. Find the equation whose roots are

$$\pm\tan\frac{\pi}{7},\quad \pm\tan\frac{2\pi}{7},\quad \pm\tan\frac{3\pi}{7}.$$

12. Prove that

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \cdots\cdots + \tan^{-1}x_n$$

$$= \tan^{-1}\frac{p_1 - p_3 + p_5 - \cdots\cdots}{1 - p_2 + p_4 - \cdots\cdots},$$

where p_r denotes the sum of the products of $x_1, x_2, \ldots\ldots\ldots x_n$ taken r at a time.

13. If x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - x^3\sin 2a + x^2\cos 2a - x\cos a - \sin a = 0,$$

show that $\Sigma\tan^{-1}x_1 = n\pi + \frac{1}{2}\pi - a$.

14. Prove that

$$\sin m\theta = \sec^m\theta\left\{m\tan\theta - \frac{m(m+1)(m+2)}{3\,!}\tan^3\theta + \cdots\right\},$$

when $\tan\theta < 1$.

[*$\cos m\theta - i\sin m\theta = (\cos\theta + i\sin\theta)^{-m}$. Expand by Binomial Theorem and equate imaginary parts.*]

15. Prove that

$$\frac{\sin^3\theta}{3\,!} = \frac{\theta^3}{3\,!} - (1+3^2)\frac{\theta^5}{5\,!} + (1+3^2+3^4)\frac{\theta^7}{7\,!} - \cdots\cdots$$

16. Show that

$$\frac{\pi^2}{2.4} - \frac{\pi^4}{2.4.6.8} + \frac{\pi^6}{2.4.6.8.10.12} - \cdots\cdots = 1.$$

17. Find the value of the series

$$1-\frac{2}{3\,!}+\frac{3}{5\,!}-\frac{4}{7\,!}+\cdots\cdots$$

18. If $x=\frac{2}{1\,!}-\frac{4}{3\,!}+\frac{6}{5\,!}-\frac{8}{7\,!}+\cdots\cdots$ to ∞,

and $y=1+\frac{2}{1\,!}-\frac{2^3}{3\,!}+\frac{2^5}{5\,!}-\cdots\cdots$ to ∞,

show that $x^2=y$.

19. If $x \cot x=a_0+a_2x^2+a_4x^4+\cdots\cdots$, show that

$$\frac{a_{2n}}{\underline{|1}}-\frac{a_{2n-2}}{\underline{|3}}+\frac{a_{2n-4}}{\underline{|5}}-\cdots\cdots+\frac{(-1)^n a_0}{\underline{|2n+1}}=\frac{(-1)^n}{\underline{|2n}}.$$

Hence, expand $x \cot x$ in ascending powers of x as far

20. If $m=v-2e \sin v+\frac{3}{4}e^2 \sin 2v$, where e is a small quantity of which all powers above the second are neglected, show that

$$v=m+2e \sin m+\tfrac{5}{4}e^2 \sin 2m.$$

21. If $\tan x=a_1x+\frac{a_3}{3\,!}x^3+\frac{a_5}{5\,!}x^5+\cdots$, show that

$$a_{2n+1}=\frac{(2n+1)2n}{2\,!}a_{2n-1}-\frac{(2n+1)2n(2n-1)(2n-2)}{4\,!}a_{2n-3}$$

$$+\cdots+(-1)^{n+1}(2n+1)a_1+(-1)^n. \quad [\textit{C. H. 1940.}]$$

22. If $\frac{1}{a}=(2n+1)\frac{\pi}{2}$, where n has any large positive integral value, prove that the large roots of the equation $\tan x=x$ are approximately $\pm\left(\frac{1}{a}-a-\frac{2}{3}a^2\right)$. [*C. H. 1941.*]

ANSWERS

1. (i) $\frac{1}{4}\left[\frac{x^3}{3!}(3^3-3)-\frac{x^5}{5!}(3^5-3)+\frac{x^7}{7!}(3^7-3)-\cdots\right]$;

$1-\frac{2x^2}{2!}(1+1)+\frac{2^3x^4}{4!}(1+4)-\frac{2^5x^6}{6!}(1+4^2)+\frac{2^7x^8}{8!}(1+4^3)-\cdot$

(ii) $\frac{1}{2}\left[\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\cdots\right]$;

$\frac{1}{4}\left[4-\frac{x^2}{2!}(3^2+3)+\frac{x^4}{4!}(3^4+3)-\cdots\right]$.

2. $\frac{1}{2\sqrt{2}}\left[1+\Sigma(-1)^{\frac{r(r-1)}{2}}\frac{(2x)^r}{r!}\right]$.

4. (ii) 0·0523. 5. (iv) 3°49′ approximately.

6. (i) $16\sin^5x-20\sin^3x+5\sin x$.

(ii) $16\cos^5\theta-20\cos^3\theta+5\cos\theta$.

10. (i) $\frac{1}{6}$. (ii) $\frac{1}{2}$. (iii) 0. (iv) 24.

11. $t^6-21t^4+35t^2-7=0$, where $t=\tan\theta$.

17. $\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}+1\right)$. 19. $1-\frac{1}{3}x^2-\frac{1}{45}x^4+\cdots\cdots$

CHAPTER IX

TRIGONOMETRICAL AND EXPONENTIAL FUNCTIONS OF COMPLEX ARGUMENTS

57. Generalised Definitions.

When x is *real*, we know what are meant by the exponential function e^x and the trigonometrical functions $\sin x$, $\cos x$, $\tan x$, etc. Now the question arises, what would be meant by these functions, when x is *complex*, *i.e.*, of the form $a+ib$. At present they have no meaning.

When x is of the form $a+ib$, let us *define* the exponential and trigonometrical functions by the following equations :—

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots\cdots$$

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\cdots$$

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots\cdots$$

Also, $$\tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{\cos x}{\sin x},$$

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sec x = \frac{1}{\cos x}.$$

Note. When x is real, it has been actually proved that e^x, $\sin x$, $\cos x$ are equal to the corresponding series ; now, when x is *complex*, for the sake of uniformity and convenience, we make the *convention* that e^x, $\sin x$, $\cos x$ should *denote* the corresponding series.

Obs. When x is **real**, e in e^x having the meaning $1+\frac{1}{1\,!}+\frac{1}{2\,!}+\cdots$, e^x has the usual significance of a certain quantity being raised to a certain power. Incidentally it has been proved to be equal to the series $1+\frac{x}{1\,!}+\frac{x^2}{2\,!}+\cdots\cdots\cdots$. But when x is **complex**, e^x has no other

significance except that it stands as a symbol or short way of writing the series $1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$

Accordingly some writers use the symbols exp. (x) or $E(x)$ in preference to e^x for the series when x is complex.

The symbol e^x however is favoured by many, because this brings about a uniformity in the two cases whether x be real or complex, since, as will be shown below, all properties (index law etc.) satisfied by e^x when x is real, will continue to be satisfied for complex values of x with the new definition of e^x.

58. Index Laws for the Exponential Function.

When x is *real* it has been proved that e^x obeys the index law. Now, it will be proved that even when x is complex, e^x with its new definition, obeys the index law.

When x and y are *complex*, we have, by the definition,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\cdots$$

$\therefore$ by multiplication, we get

$$e^x.e^y = 1 + (x+y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!}\right) + \cdots\cdots$$

$$+ \left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!}\cdot\frac{y}{1} + \cdots + \frac{x}{1}\cdot\frac{y^{n-1}}{(n-1)!} + \frac{y^n}{n!}\right) + \cdots\cdots$$

$$= 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \cdots + \frac{1}{n!}(x+y)^n + \cdots$$

$$= e^{x+y}, \text{ by definition*}.$$

Similarly, $e^x.e^y.e^z\cdots\cdots = e^{x+y+z+\cdots\cdots}$

It also easily follows that $e^x \div e^y = e^{x-y}$

and $(e^x)^n = e^{nx}$, n being a positive integer.

The last result can be extended to the case where n is any real quantity, e^{nx} being then one of the values of $(e^x)^n$.

*For convergence consideration, see *Appendix*, § 6(1).

59. Exponential values of sine and cosine.

If we put ix for z in the equation

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\cdots$$

we obtain, $$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \cdots\right),$$

i.e., $\mathbf{e^{ix} = \cos x + i \sin x}$,* $\cdots$ (1)

whether x be real or complex.

Similarly, $\mathbf{e^{-ix} = \cos x - i \sin x}$. $\cdots$ (2)

Hence, for all values of x, *real* or *complex*, we have from (1) and (2),

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \text{ and } \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \cdots \quad (3)$$

These expressions for $\cos x$ and $\sin x$ are known as *Euler's Exponential values.*

When x is a complex quantity, *sin x, cos x are sometimes defined by the relations* (3), e^x being first defined as in Art. 57.

60. Addition and Subtraction Theorems (*for complex arguments*).

When x and y are *complex*, to prove that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

*This result is known as **Euler's Theorem**, after the name of the great Swiss mathematician *Euler* (1707-1783).

8

Proof. By Art. 58, we have

$$e^{i(x+y)} = e^{ix}.e^{iy};$$

therefore, by Art. 59,

$$\begin{aligned}\cos(x+y) + i\sin(x+y) &= (\cos x + i\sin x)(\cos y + i\sin y)\\ &= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y) \quad \cdots (1)\end{aligned}$$

Again, from the identity,

$$e^{-i(x+y)} = e^{-ix}.e^{-iy},$$

we have, as before

$$\begin{aligned}\cos(x+y) - i\sin(x+y) &= (\cos x - i\sin x)(\cos y - i\sin y)\\ &= \cos x\cos y - \sin x\sin y\\ &\qquad - i(\sin x\cos y + \cos x\sin y) \quad \cdots (2)\end{aligned}$$

Now, adding and subtracting (1) and (2), the required values of $\sin(x+y)$ and $\cos(x+y)$ are obtained.

Again, since, $e^{i(x-y)} = e^{ix}.e^{-iy}$,

$$\begin{aligned}\therefore \quad \cos(x-y) + i\sin(x-y) &= (\cos x + i\sin x)(\cos y - i\sin y)\\ &= \cos x\cos y + \sin x\sin y\\ &\qquad + i(\sin x\cos y - \cos x\sin y) \quad \cdots (3)\end{aligned}$$

and since, $e^{-i(x-y)} = e^{-ix}.e^{iy}$,

$$\begin{aligned}\therefore \quad \cos(x-y) - i\sin(x-y) &= (\cos x - i\sin x)(\cos y + i\sin y)\\ &= \cos x\cos y + \sin x\sin y\\ &\qquad - i(\sin x\cos y - \cos x\sin y) \quad \cdots (4)\end{aligned}$$

Now, adding and subtracting (3) and (4), the required values of $\sin(x-y)$ and $\cos(x-y)$ are obtained.

Alternative Method. The above formulæ can also be established by making use of the exponential values of $\sin x$, $\sin y$, and $\cos x$, $\cos y$; thus,

$$\sin x \cos y + \cos x \sin y$$

$$= \frac{1}{2i}(e^{ix} - e^{-ix}) \cdot \frac{1}{2}(e^{iy} + e^{-iy})$$

$$+ \frac{1}{2}(e^{ix} + e^{-ix}) \cdot \frac{1}{2i}(e^{iy} - e^{-iy})$$

$$= \frac{e^{ix}.2e^{iy} - e^{-ix}.2e^{-iy}}{4i} = \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i}$$

$$= \sin(x+y).$$

Similarly, the other relations can be proved.

61. From the above article, it follows that all trigonometrical formulæ involving real angles which were deduced from Addition and Subtraction theorems are also true when complex quantities are substituted for real angles ; thus, whether x be *real* or *complex*, we have

$$\sin 2x = 2 \sin x \cos x \; ; \; \cos 2x = \cos^2 x - \sin^2 x \; ; \text{ etc.}$$

62. De Moivre's Theorem for Complex Arguments.

Whether θ be *real* or *complex*, we have

$$(e^{i\theta})^n = e^{in\theta}.$$

$$\therefore \quad (\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta.$$

Thus, De Moivre's Theorem is true, whether θ be real or complex.

63. Periodicity of Circular and Exponential functions.

In the first and second equations of Art. 60, put $y = 2n\pi$, where n is any integer and let x be complex ; then

$$\sin(x + 2n\pi) = \sin x \cos 2n\pi + \cos x \sin 2n\pi$$

$$= \sin x.$$

Similarly, $\cos(x+2n\pi)=\cos x$.

Thus, even when x is *complex*, $\sin x$ and $\cos x$ are *periodic functions.*

Again, $e^{i(2n\pi+\theta)}=\cos(2n\pi+\theta)+i\sin(2n\pi+\theta)$

$$=\cos\theta+i\sin\theta=e^{i\theta}.$$

Thus, $e^{i\theta}$ is a *periodic* function; that is, its value remains unchanged, when θ is increased by any multiple of 2π.

Also, $\mathbf{e^{2n\pi i}}=\cos 2n\pi+i\sin 2n\pi=\mathbf{1}$.

64. Logarithm of a Complex Quantity ; a many-valued function.

If z and A be two complex quantities, and if they be so related that $e^z=A$,

i.e., $1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\cdots=A$,

then, z is said to be a *logarithm of* A.

Again, if $e^z=A$,

since, $e^{2n\pi i}=1$, where n is zero, or any integer,

we have,

$$A=e^z.e^{2n\pi i}=e^{z+2n\pi i}.$$

Hence, it follows from the above definition that if z be a logarithm of A, so also is $z+2n\pi i$.

Thus, we see that *the logarithm of a complex quantity has an infinite number of values.* When this many-valuedness is taken into consideration, logarithm of A is sometimes written as Log A in order to distinguish it from its principal value which is written as $\log A$. The *principal value* of the logarithm is obtained by putting $n=0$ in the value of Log A. Thus, we have

$$\text{Log } A=2n\pi i+\log A.$$

Note. If s and A be both real, then since the above proof equally holds, we get logarithm of $A = s + 2n\pi i$; that is, *a real quantity has one real logarithm and an infinite number of imaginary logarithms.*

65. Definition of a^x (*when a and x are complex quantities*).

When a and x are real quantities, we know that $a^x = e^{x \log_e a}$. When a and x are complex, the ordinary definition of a^x no longer holds.

When a and x are *complex quantities*, let us define a^x by the equation

$$a^x = e^{x \operatorname{Log} a}$$

Since, Log a is multiple-valued and complex when a is complex, therefore, a^x is multiple-valued and complex and

$$a^x = e^{x \operatorname{Log} a} = e^{x(2n\pi i + \log a)}.$$

The *principal value of a^x* (which is the value of a^x obtained by putting $n = 0$)

$$= e^{x \log a}$$

$$= 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \cdots\cdots$$

As a special case, putting $x = 1$,

$$a = e^{\log a} = 1 + \log a + \frac{1}{2!}(\log a)^2 + \cdots\cdots$$

66. From the definition of the logarithm of a complex quantity, it can be easily shown that when x and y are both complex quantities,

$$\operatorname{Log} xy = \operatorname{Log} x + \operatorname{Log} y$$

$$\operatorname{Log} \frac{x}{y} = \operatorname{Log} x - \operatorname{Log} y$$

$\operatorname{Log} x^y = y \operatorname{Log} x + 2k\pi i$, where k is zero, or any integer.

For, let $e^a = x$ and $e^b = y$, so that $xy = e^a . e^b = e^{a+b}$.

Then $\text{Log } x = 2m\pi i + a$, $\text{Log } y = 2n\pi i + b$.

$$\therefore \quad \text{Log } x + \text{Log } y = 2(m+n)\pi i + a + b = 2k\pi i + a + b$$
$$= \text{Log } xy.$$

Similarly for other relations.

67. Inverse circular functions of Complex Arguments.

If $\cos(x+iy) = u + iv$, then $x+iy$ is defined as an inverse cosine of $u+iv$ and is written as $x + iy = \cos^{-1}(u+iv)$.

Since, $\cos(x+iy) = \cos\{2n\pi \pm (x+iy)\}$.

$\therefore \quad \cos\{2n\pi \pm (x+iy)\} = u + iv$.

Hence, $2n\pi \pm (x+iy)$ is also an inverse cosine of $u+iv$ (*n being any integer*).

Thus, the inverse cosine of $u+iv$ is a many-valued function and when this many-valuedness is taken into consideration, it is written $\text{Cos}^{-1}(u+iv)$.

The *principal value** of $\text{Cos}^{-1}(u+iv)$ is defined as that value for which the real part lies between 0 and π, and is denoted by $\cos^{-1}(u+iv)$.

We have, then

$$\text{Cos}^{-1}(u+iv) = 2n\pi \pm \cos^{-1}(u+iv).$$

Similarly, other inverse circular functions and their principal values are defined.

Thus, if $\sin(x+iy) = u + iv$,

then $\quad u + iv = \sin\{n\pi + (-1)^n(x+iy)\}$,

$$\therefore \quad \text{Sin}^{-1}(u+iv) = n\pi + (-1)^n(x+iy)$$
$$= n\pi + (-1)^n \sin^{-1}(u+iv).$$

* This definition agrees with that for the principal value of $\cos^{-1}a$, when a is real.

If $\quad \tan(x+iy) = u+iv,$

then, $\quad u+iv = \tan\{n\pi + x + iy\}.$

$$\therefore \quad \text{Tan}^{-1}(u+iv) = n\pi + (x+iy)$$
$$= n\pi + \tan^{-1}(u+iv).$$

The *principal values* of $\text{Sin}^{-1}(u+iv)$ and $\text{Tan}^{-1}(u+iv)$ are in each case those values for which the real part lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Similarly, $\text{Sec}^{-1}(u+iv) = 2n\pi \pm \sec^{-1}(u+iv)$

$$\text{Cosec}^{-1}(u+iv) = n\pi + (-1)^n \operatorname{cosec}^{-1}(u+iv)$$
$$\text{Cot}^{-1}(u+iv) = n\pi + \cot^{-1}(u+iv).$$

68. Binomial Theorem for a Complex Quantity.*

If z be *real* and *numerically less than* 1, then for all *real values of n*, we have the Binomial Theorem

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2\,!} z^2 + \frac{n(n-1)(n-2)}{3\,!} z^3 + \cdots\cdots(1)$$

When z is *complex* (say $= a+ib$) then,

(i) if n *be a positive integer,* the series on the right side of (1) is finite for all values of z and hence the Binomial Theorem is true for all values of z ;

(ii) if n *be negative or fractional,* the number of terms of the series is infinite and the series is equal to *one of* the values of $(1+z)^n$ and it is generally taken to be the *principal value* of $(1+z)^n$, provided *the modulus of z,* that is $+\sqrt{a^2+b^2}$, *is numerically less than* 1 ;

*Here the theorem with its various restrictions is simply stated, for the sake of reference, without any proof, as the proof is difficult and beyond the range of the present treatise. For the proof, the student may refer to Knopp's *Theory and Applications of Infinite Series* or Chrystal's *Algebra, Vol. II.*

(iii) if *n be negative or fractional and if the modulus of* z is equal to 1, then the theorem is true, when

(*a*) n is a positive fraction, or

(*b*) n is a negative proper fraction, *i.e.*, when n lies between 0 and -1, and z is not equal to -1.

Note. The theorem will *not* hold

(i) when mod. $z=1$ and $n=-1$ or <-1

and also (ii) when mod. $z>1$ and n is negative or fractional.

69. Logarithmic series for a Complex Quantity.

When z is *real* and *less than* or *equal to* 1, we know that

$$\log(1+z)=z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\cdots \quad \cdots \quad (1)$$

When z is *complex*, it can be shown exactly in the same way as when z is real, that the series on the right side of (1) is the expansion of $\log(1+z)$, *i.e.*, *the principal value of* $\text{Log}(1+z)$, provided

(i) *mod. z is numerically less than* 1

and (ii) if *mod.* $z=1$, then also the theorem is true if the *amplitude of z be not equal to an odd multiple of π.*

Since, $\text{Log}(1+z)=2n\pi i+\log(1+z)$,

therefore, $\text{Log}(1+z)=2n\pi i+z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\cdots\cdots$

70. Ex. 1. *Evaluate $\text{Log}(\alpha+i\beta)$, where α and β are real.*

[*C. H. 1942.*]

Let $\text{Log}(\alpha+i\beta)=x+iy$ (x and y being real);

then, $e^{x+iy}=\alpha+i\beta$.

Put $\alpha=r\cos\theta$, $\beta=r\sin\theta$, so that $r=+\sqrt{\alpha^2+\beta^2}$, $\tan\theta=\dfrac{\beta}{\alpha}$;

then $\alpha+i\beta=r(\cos\theta+i\sin\theta)$

$$=re^{i\theta}=e^{\log r}.e^{i\theta}.e^{2n\pi i}=e^{\log r+i(2n\pi+\theta)}.$$

Therefore, $e^{x+iy}=e^{\log r+i(2n\pi+\theta)}$.

Hence, $x+iy=\log r+i(2n\pi+\theta)$.

Equating real and imaginary parts, we have,

$$x=\log r \text{ and } y=2n\pi+\theta.$$

$\therefore$ $\operatorname{Log}(\alpha+i\beta)=\log r+i(2n\pi+\theta)$

$$=\log_e\sqrt{\alpha^2+\beta^2}+i\left\{2n\pi+\tan^{-1}\frac{\beta}{\alpha}\right\}.$$

Note. Putting $n=0$, in the above result, we get the principal value of $\operatorname{Log}(\alpha+i\beta)$;

i.e., $\log(\alpha+i\beta)=\log_e\sqrt{\alpha^2+\beta^2}+i\tan^{-1}\dfrac{\beta}{\alpha}$.

Ex. 2. *Separate into its real and imaginary parts the expression* $(\alpha+i\beta)^{x+iy}$. [*C. P.* 1930, '32, '44]

By definition in Art. 65,

$$(\alpha+i\beta)^{x+iy}=e^{(x+iy)\operatorname{Log}(\alpha+i\beta)}.$$

Now, we have as in Ex. 1,

$$\operatorname{Log}(\alpha+i\beta)=\log r+i\,(2n\pi+\theta),$$

where $r=\sqrt{\alpha^2+\beta^2}$ and $\theta=\tan^{-1}\dfrac{\beta}{\alpha}$.

Hence, $(x+iy)\operatorname{Log}(\alpha+i\beta)$

$$=x\log r-y(2n\pi+\theta)+i\{y\log r+x\,(2n\pi+\theta)\}$$

$$=p+iq, \text{ say.}$$

Then,

$$(\alpha+i\beta)^{x+iy}=e^{p+iq}=e^p.e^{iq}$$

$$=e^p(\cos q+i\sin q)$$

$$=e^{x\log r-y(2n\pi+\theta)}[\cos\{y\log r+x(2n\pi+\theta)\}$$

$$+i\sin\{y\log r+x(2n\pi+\theta)\}].$$

Note. The *principal value* of the given expression, obtained by putting $n=0$ in the above result is

$$e^{x \log r - y\theta}\ [\cos(y \log r + x\theta) + i \sin(y \log r + x\theta)].$$

Ex. 3. *Prove that* $i^i = e^{-(4n+1)\frac{\pi}{2}}$. [*C. H. 1932* , *C. P. 1942.*]

We have, $i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

$$= e^{i\frac{\pi}{2}} . e^{2n\pi i} = e^{i\frac{\pi}{2} + 2n\pi i} = e^{i\frac{\pi}{2}(4n+1)}.$$

$\therefore$ $\quad \text{Log}\, i = \log e^{i\frac{\pi}{2}(4n+1)} = i\frac{\pi}{2}(4n+1).$

$\therefore$ $\quad i^i = e^{i \,\text{Log}\, i} = e^{i.i.\frac{\pi}{2}(4n+1)} = e^{-\frac{\pi}{2}(4n+1)}.$

Ex. 4. *Prove that* $\text{Sin}^{-1}(i) = 2n\pi - i \log(\sqrt{2}-1)$.

Let $\quad \text{Sin}^{-1} i = \theta. \quad \therefore \quad \sin\theta = i.$

$\therefore$ $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-i^2} = \sqrt{1 \mp 1} = \sqrt{2}.$

$\therefore$ $\quad e^{i\theta} = \cos\theta + i \sin\theta = \sqrt{2} - 1.$

$\therefore$ $\quad i\theta = 2n\pi i + \log(\sqrt{2}-1)$

$\therefore$ $\quad \theta$ or $\text{Sin}^{-1}(i) = 2n\pi - i \log(\sqrt{2}-1).$

Ex. 5. *If* $\tan \log(x+iy) = a+ib$, *where* $a^2+b^2 \neq 1$ *prove that*

$$\tan \log(x^2+y^2) = \frac{2a}{1-a^2-b^2}. \qquad [\ C.\ P.\ 1946.\ ]$$

Here, $\quad \log(x+iy) = \tan^{-1}(a+ib),$

$\therefore$ $\log(x-iy) = \tan^{-1}(a-ib),$

$\therefore$ $\log(x^2+y^2) = \log\{(x+iy)(x-iy)\} = \log(x+iy) + \log(x-iy)$

$$= \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$$

$$= \tan^{-1}\frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)} = \tan^{-1}\frac{2a}{1-(a^2+b^2)}.$$

$\therefore$ $\tan\log(x^2+y^2) = \dfrac{2a}{1-a^2-b^2}$

Ex. 6. *Find all the values of z if $i^z = e\,(\cos\theta + i\sin\theta)$.*

Since, $\quad i^z = e^{z \text{ Log } i}$,

$$\therefore \quad e^{z \text{ Log } i} = e.e^{i\theta} = e.e^{2n\pi i}.e^{i\theta} = e^{1+i(2n\pi+\theta)}.$$

$$\therefore \quad z \text{ Log } i = 1 + i\,(2n\pi + \theta).$$

Since, $\quad \text{Log } i = i\frac{\pi}{2}(4k+1)$, [see *Ex. 3, above*]

$$\therefore \quad z.i\frac{\pi}{2}(4k+1) = 1 + i(2n\pi + \theta).$$

$$\therefore \quad z = \frac{2}{(4k+1)\pi}(2n\pi + \theta - i), \text{ putting } 1 = -i^2.$$

Ex. 7. *If $x = \log\tan\left(\frac{\pi}{4} + \frac{y}{2}\right)$, prove that*

$$y = -i \text{ Log } \tan\left(\frac{ix}{2} + \frac{\pi}{4}\right).$$

Here, $\quad e^x = \tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \dfrac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}}, \qquad \tan\frac{y}{2} = \dfrac{e^x - 1}{e^x + 1},$

$$\text{or,} \quad \frac{e^{\frac{iy}{2}} - e^{-\frac{iy}{2}}}{i\left(e^{\frac{iy}{2}} + e^{-\frac{iy}{2}}\right)} = \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} = -\frac{e^{i\frac{ix}{2}} - e^{-i\frac{ix}{2}}}{e^{i\frac{ix}{2}} + e^{-i\frac{ix}{2}}} = -i\tan\frac{ix}{2}.$$

$$\therefore \quad \frac{e^{\frac{iy}{2}} - e^{-\frac{iy}{2}}}{e^{\frac{iy}{2}} + e^{-\frac{iy}{2}}} = -i^2\tan\frac{ix}{2} = \tan\frac{ix}{2}.$$

$$\therefore \quad e^{iy} = \frac{1+\tan\frac{ix}{2}}{1-\tan\frac{ix}{2}} = \tan\left(\frac{\pi}{4} + \frac{ix}{2}\right),$$

$$\therefore \quad iy = \text{Log } \tan\left(\frac{ix}{2} + \frac{\pi}{4}\right) = -i^2 \text{ Log } \tan\left(\frac{ix}{2} + \frac{\pi}{4}\right),$$

$$\text{or,} \quad y = -i \text{ Log } \tan\left(\frac{ix}{2} + \frac{\pi}{4}\right).$$

Ex. 8. *Express Log {Log (cos θ + i sin θ)} in the form A + iB.*
[*C. H. 1944, '51.*]

$$\text{Log}(\cos\theta + i\sin\theta) = \text{Log}\, e^{i\theta} = \log e^{i(2n\pi+\theta)} = (2n\pi+\theta)i$$

$$= (2n\pi+\theta)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = (2n\pi+\theta)e^{i\frac{\pi}{2}}.$$

$\therefore$ Log {Log (cos $\theta + i$ sin θ)}

$$= \text{Log}\left\{(2n\pi+\theta).e^{i\frac{\pi}{2}}\right\}$$

$$= \log(2n\pi+\theta) + \text{Log}\, e^{i\frac{\pi}{2}}$$

$$= \log(2n\pi+\theta) + 2k\pi i + i\frac{\pi}{2}$$

$$= \log(2n\pi+\theta) + i\pi(2k+\tfrac{1}{2}).$$

EXAMPLES IX

[*All letters represent real numbers unless otherwise stated*]

1. From the definitions of sin x and cos x where x is a complex quantity, prove that

(i) $\cos 2x = 1 - 2\sin^2 x$.

(ii) $\sin 3x = 3\sin x - 4\sin^3 x$. [*C. P. 1943.*]

(iii) $\sin^2 x + \cos^2 x = 1$.

(iv) $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

2. Apply the exponential values of sine and cosine to show that

(i) $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$.

(ii) $\cos 2\theta = 1 - 2\sin^2\theta$. [*C. P. 1925.*]

3. Show that

(i) Log $(-1) = (2n+1)i\pi$. (ii) L

(iii) Log $(-x) = \log x + (2n+1)i\pi$.

(iv) $\text{Log}\,(1+i) = \frac{1}{2}\log 2 + (2n+\frac{1}{4})i\pi$.

(v) $\tan\left\{i \log \dfrac{a-ib}{a+ib}\right\} = \dfrac{2ab}{a^2-b^2}$. [*C. P. 1942, '48.*]

(vi) $x^i = e^{-2n\pi}\{\cos(\log x) + i \sin(\log x)\}$.

(vii) $i^x = \cos k\pi x + i \sin k\pi x$, where $k = 2n + \frac{1}{2}$.

(viii) $\text{Log}_i i = \dfrac{4m+1}{4n+1}$, where m and n are any integers.

(ix) $\log(1+\cos 2\theta + i \sin 2\theta) = \log(2\cos\theta) + i\theta$.
[*C. P. 1947.*]

(x) $\text{Sin}^{-1}(ix) = 2n\pi - i \log(\sqrt{1+x^2} - x)$

$= 2n\pi + i \log(\sqrt{1+x^2} + x)$.

(xi) $\tan^{-1}(e^{i\theta}) = \frac{1}{4}\pi + \frac{1}{2}i \log \tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$.

(xii) $\sin(\log i^i) = -1$.

(xiii) $(\sin x + i \cos x)^i = e^{x-2n\pi-\frac{1}{2}\pi}$.

(xiv) $(-i)^{-i} = e^{\frac{1}{2}(4n-1)\pi}$. (xv) $\text{Log}(\sqrt{i}) = (2n\pi + \frac{1}{4}\pi)i$.

(xvi) $\text{Log}_{10} 2 = \dfrac{\log 2 + 2m\pi i}{\log 10 + 2n\pi i}$.

4. If $A + iB = \log(x+iy)$, show that

$$B = \tan^{-1}\frac{y}{x} \text{ and } A = \tfrac{1}{2}\log(x^2+y^2).$$

5. If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

6. If $i^{i^{i \dots \text{ad. inf.}}} = A + iB$, prove that

$$\tan\frac{\pi A}{2} = \frac{B}{A} \text{ and } A^2 + B^2 = e^{-\pi B}.$$ [. . . . .]

7. Express in the form $A+iB$

(i) π^i. (ii) $\text{Log}\,(-e)$.

(iii) $\cos\,(a+i\beta)$. [*C. P. 1931.*]

(iv) $\log \sin\,(\theta+i\phi)$.

(v) $(x+iy)^i$. (vi) $\text{Log}\, e^{x+iy}$.

8. Show that

$$\log \log\,(x+iy)=\tfrac{1}{2}\log\,(p^2+q^2)+i\tan^{-1}\frac{q}{p},$$

where $p=\log\sqrt{x^2+y^2}$ and $q=\tan^{-1}\dfrac{y}{x}$.

9. Prove that the principal value of $(a+i\beta)^{x+iy}$ is wholly real or wholly imaginary according as

$$\tfrac{1}{2}y\log\,(a^2+\beta^2)+x\tan^{-1}\frac{\beta}{a}$$

is an even or odd multiple of $\frac{1}{2}\pi$.

10. If $\tan\,(x+iy)=u+iv$, prove that

$$u^2+v^2+2u\cot 2x=1. \qquad [\ \textit{C. P. 1945, '48.}\ ]$$

11. If z be a complex quantity, show that

$$\tan z=z+\tfrac{1}{3}z^3+\tfrac{2}{15}z^5+\cdots$$

12. If z be a complex quantity, prove that

(i) the zeros of $\sin z$ are given by $z=k\pi$

and (ii) the zeros of $\cos z$ are given by $z=(2k+1)\frac{1}{2}\pi$,

where k is zero or any integer.

13. Show that the values of i^i can be arranged so that they form a geometrical progression. [*C. H. 1951.*]

14. Prove that

$$\log\frac{a+ib-x}{a+ib+x}=\frac{1}{2}\log\frac{(a-x)^2+b^2}{(a+x)^2+b^2}+i\left\{\tan^{-1}\frac{b}{a-x}-\tan^{-1}\frac{b}{a+x}\right\}.$$

15. (i) Prove that the principal value of

$$i^{\log(1+i)}=e^{-\frac{\pi^2}{8}}\{\cos(\tfrac{1}{4}\pi\log 2)+i\sin(\tfrac{1}{4}\pi\log 2)\}.$$

(ii) Find all the values of $(1+i)^{1+i}$.

16. Prove that the ratio of the principal values of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is $\sin(\log 2)+i\cos(\log 2)$.

17. (i) Solve $\cos x=c$, $c>1$.

(ii) If $a\cos\theta+b\sin\theta=c$, $c>\sqrt{a^2+b^2}$, find the general value of θ.

18. If x be $<\frac{1}{2}\pi$, show that $i\cos^{-1}(\sin x+\cos x)$ has two real values.

19. If $\tan^{-1}(a+ib)=\sin^{-1}(x+iy)$, show that

$$a^2+b^2=\frac{x^2+y^2}{\sqrt{x^4+y^4+2x^2y^2-2x^2+2y^2+1}}.$$

20. If $k=\begin{vmatrix} e^{i\alpha} & e^{-i\alpha} & 1 \\ e^{i\beta} & e^{-i\beta} & 1 \\ e^{i\gamma} & e^{-i\gamma} & 1 \end{vmatrix}$

show that

$$k^2=\begin{vmatrix} 2\cos 2\alpha+1 & 2\cos(\alpha+\beta)+1 & 2\cos(\alpha+\gamma)+1 \\ 2\cos(\beta+\alpha)+1 & 2\cos 2\beta+1 & 2\cos(\beta+\gamma)+1 \\ 2\cos(\gamma+\alpha)+1 & 2\cos(\gamma+\beta)+1 & 2\cos 2\gamma+1 \end{vmatrix}$$

[*C. H. 1927.*]

ANSWERS

7. (i) $e^{-2n\pi}\{\cos(\log\pi)+i\sin(\log\pi)\}$. (ii) $1+(2n+1)\pi i$.

(iii) $\cos a.\frac{1}{2}\left(e^{-\beta}+e^{\beta}\right)+i\sin a.\frac{1}{2}\left(e^{-\beta}-e^{\beta}\right)$.

(iv) $\frac{1}{2}\log\frac{1}{4}(e^{2\phi}+e^{-2\phi}-2\cos 2\theta)+i\tan^{-1}\left\{\cot\theta.\frac{e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}}\right\}$.

(v) $e^{-\left(2n\pi+\tan^{-1}\frac{y}{x}\right)}[\cos\{\frac{1}{2}\log(x^2+y^2)\}+i\sin\{\frac{1}{2}\log(x^2+y^2)\}]$.

(vi) $x+i(y+2n\pi)$.

15. (ii) $2^{\frac{1}{2}}e^{-(2n+\frac{1}{4})\pi}[\cos\{\frac{1}{2}\log 2+(2n+\frac{1}{4})\pi\}+i\sin\{\frac{1}{2}\log 2+(2n+\frac{1}{4})\pi\}]$.

17. (i) $2n\pi\pm i\log(c+\sqrt{c^2-1})$. [where n is any integer, positive or negative.]

(ii) $\theta=2n\pi+i\log\frac{c+\sqrt{c^2-a^2-b^2}}{\sqrt{a^2+b^2}}+\tan^{-1}\frac{b}{a}$.

CHAPTER X

GREGORY'S SERIES AND EVALUATION OF π

71. Gregory's Series.

To expand θ in powers of $\tan\theta$. $(-\frac{1}{4}\pi \leqslant \theta \leqslant +\frac{1}{4}\pi)$.

We have, $i\tan\theta = \dfrac{i\sin\theta}{\cos\theta} = \dfrac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$.

$\therefore$ by componendo and dividendo, we have

$$\frac{1+i\tan\theta}{1-i\tan\theta} = \frac{e^{i\theta}}{e^{-i\theta}} = e^{2i\theta}.$$

Taking logarithms of both sides (considering the principal value only)

$$\begin{aligned} 2i\theta &= \log(1+i\tan\theta) - \log(1-i\tan\theta) \\ &= i\tan\theta - \tfrac{1}{2}i^2\tan^2\theta + \tfrac{1}{3}i^3\tan^3\theta - \cdots\cdots \\ &\quad -(-i\tan\theta - \tfrac{1}{2}i^2\tan^2\theta - \tfrac{1}{3}i^3\tan^3\theta - \cdots\cdots) \end{aligned}$$

by Art. 69, if $\tan\theta$ be not numerically greater than 1, *i.e.*, if $-\frac{1}{4}\pi \leqslant \theta \leqslant +\frac{1}{4}\pi$ (the principal value of θ being considered)

$$= 2i\,\{\tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots\cdots\}.$$

$$\therefore\quad \boldsymbol{\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots \text{ad. inf.}} \cdots \qquad (1)$$

where $\quad -\frac{1}{4}\pi \leqslant \theta \leqslant +\frac{1}{4}\pi$.

This is called **Gregory's Series.***

Putting $\tan\theta = x$, so that $\theta = \tan^{-1}x$, the above series may be written as

$$\mathbf{\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots} \qquad (2)$$

where $-1 \leqslant x \leqslant +1$ and $\tan^{-1}x$ has its principal value.

---

*This series was first given by *James Gregory* (1638-1675), Professor at Edinburgh, for obtaining the approximate value of π.

Note 1. In considering the principal values of both sides, the principal value of θ also, consequent on the value of $\tan\theta$, is to be taken into account.

It is also to be noted that if $\tan\theta$ be numerically not greater than 1, the principal value of θ from definition lies between $+\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$.

Note 2. If x be *complex*, then also we have

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots\cdots$$

provided the modulus of x is less than 1 and the real part of $\tan^{-1}x$ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

For the proof see Chap. XI, Bromwich's *Theory of Infinite Series.*

72. *If* $n\pi - \frac{1}{4}\pi \leqslant \theta \leqslant n\pi + \frac{1}{4}\pi$, *prove that*

$$\theta - n\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots\cdots$$

Let $\theta - n\pi = \phi$, so that $-\frac{1}{4}\pi \leqslant \phi \leqslant +\frac{1}{4}\pi$

and $\tan\theta = \tan(n\pi + \phi) = \tan\phi$.

Hence, as in the previous article,

$$\phi = \tan\phi - \tfrac{1}{3}\tan^3\phi + \tfrac{1}{5}\tan^5\phi - \cdots\cdots$$

$$\therefore\quad \theta - n\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots\cdots \qquad (3)$$

Obs. This series may be considered as more general than Gregory's series, since it is applicable to the case when θ does not lie between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$. Gregory's series is obtained from this series by putting $n=0$.

Note 1. *Examples of particular cases.*

(i) If θ lies between $\frac{7}{4}\pi$ and $\frac{9}{4}\pi$, *i.e.*, between $2\pi - \frac{1}{4}\pi$ and $2\pi + \frac{1}{4}\pi$ the relation (3) becomes

$$\theta - 2\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots\cdots$$

(ii) If θ lies between $-\frac{5}{4}\pi$ and $-\frac{3}{4}\pi$, *i.e.*, between $-\pi - \frac{1}{4}\pi$ and $-\pi + \frac{1}{4}\pi$, we have $n = -1$, and the relation (3) becomes

$$\theta + \pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots\cdots$$

Note 2. If θ lies between $n\pi+\frac{1}{4}\pi$ and $n\pi+\frac{3}{4}\pi$, $\tan\theta$ is numerically greater than 1 and hence the expansions of $\log(1\pm i\tan\theta)$ do not hold and consequently there is no such relation as (3) of Art. 72.

73. Evaluation of π.

Different mathematicians have used Gregory's series in different ways for the calculation of the value of π.

Putting $\theta=\frac{1}{4}\pi$, in Gregory's series, we have

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\ldots$$

This series may be used to calculate the value of π; but as it is very slowly convergent, a very large number of terms would have to be taken to obtain the value of π correct to any great degree of accuracy. Hence, different mathematicians have given different series.

(i) Euler's Series.

$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}}$$

$$=\tan^{-1}1=\frac{\pi}{4}.$$

Hence, putting successively $\frac{1}{2}$ and $\frac{1}{3}$ for x in the series (2), we have

$$\frac{\pi}{4}=\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}$$

$$=\left\{\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}-\cdots\right\}+\left\{\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\cdots\right\}.$$

(ii) Machin's Series.*

Since, $2\tan^{-1}\frac{1}{5}=\tan^{-1}\dfrac{\frac{2}{5}}{1-\frac{1}{25}}=\tan^{-1}\frac{5}{12}$,

and $\quad 4\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{5}{12}=\tan^{-1}\dfrac{\frac{10}{12}}{1-\frac{25}{144}}$

$\tan^{-1}\frac{120}{119}$, as before.

*In 1706, *J. Machin* (1680-1751), Professor of Astronomy at Gresham College, London, obtained the value of π up to 100 places of decimals by using the above series.

$$\therefore \quad 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}$$
$$= \tan^{-1}\tfrac{120}{119} - \tan^{-1}\tfrac{1}{239}$$
$$= \frac{\pi}{4}.$$

Now, substituting successively, $\frac{1}{5}$, $\frac{1}{239}$ for x in the series (2) the *Machin's series* for $\frac{\pi}{4}$ is obtained.

(iii) Dase's Series.

Since, $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{7}{9}$, [*by Art. 23.*]

$$\therefore \quad \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5} + \tan^{-1}\tfrac{1}{8} = \tan^{-1}\tfrac{7}{9} + \tan^{-1}\tfrac{1}{8}$$
$$= \tan^{-1}1 = \frac{\pi}{4}.$$

Now, substituting $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{8}$ for x successively in the series (2), the *Dase's series* for $\frac{\pi}{4}$ is obtained.

(iv) Rutherford's Series.

$$\tan^{-1}\tfrac{1}{70} - \tan^{-1}\tfrac{1}{99} = \tan^{-1}\frac{\frac{1}{70} - \frac{1}{99}}{1 + \frac{1}{70}\cdot\frac{1}{99}} = \tan^{-1}\tfrac{1}{239}.$$

Since, $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, [see *(ii)*]

$$\therefore \quad \frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99}.$$

Now, substituting $\frac{1}{5}$, $\frac{1}{70}$, $\frac{1}{99}$ successively for x in the series (2), the *Rutherford's series* for $\frac{1}{4}\pi$ is obtained.

74. Ex. 1. *Find the numerical value of π to 5 places of decimals by Machin's series.*

We have,

$$\frac{\pi}{4} = 4\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \cdots\right\} - \left\{\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \cdots\right\}$$
$$= 4\{\cdot 2 - \cdot 0026666 + \cdot 000064 - \cdots\} - \cdot 0041841 + \cdots$$
$$\doteq \cdot 7853983\,;$$
$$\therefore \quad \pi = 4 \times \cdot 7853983 = 3\cdot 14159.$$

Ex. 2. *Prove that*

$$\frac{\pi}{8}=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\cdots \qquad [\textit{ C. P. 1924.}]$$

Putting $\theta=\frac{1}{4}\pi$ in Gregory's series and noting that $\tan\frac{1}{4}\pi=1$, we have

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$$

$$=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots$$

$$=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\cdots\cdots$$

Hence, $\dfrac{\pi}{8}=\dfrac{1}{1.3}+\dfrac{1}{5.7}+\dfrac{1}{9.11}+\cdots\cdots$

Ex. 3. *Expand* $\tan^{-1}\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$ *as a power series in* $\tan\theta$.

Given expression $=\tan^{-1}\dfrac{1+\tan\theta}{1-\tan\theta}$, on dividing numerator and denominator by $\cos\theta$,

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]=n\pi+\frac{\pi}{4}+\theta$$

$$=(n+\tfrac{1}{4})\pi+\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\cdots\cdots$$

Ex. 4. *If* $\sin x=n\sin(a+x)$, $n<1$, *expand* x *in a series of ascending powers of* n. [*C. P. 1931, '44.*]

Since, $\sin x=n\sin(a+x)=n(\sin a\cos x+\cos a\sin x)$,

$$\therefore\quad \tan x=\frac{n\sin a}{1-n\cos a}, \text{ or, } \frac{\sin x}{\cos x}=\frac{n\sin a}{1-n\cos a};$$

$$\frac{\frac{1}{2i}(e^{ix}-e^{-ix})}{\frac{1}{2}(e^{ix}+e^{-ix})}=\frac{n\sin a}{1-n\cos a};$$

$$\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}=\frac{ni\sin a}{1-n\cos a};$$

$\therefore$ by componendo and dividendo,

$$\frac{e^{ix}}{e^{-ix}}=\frac{1-n\cos a+ni\sin a}{1-n\cos a-ni\sin a}=\frac{1-ne^{-ia}}{1-ne^{ia}},$$

or, $$e^{2ix}=\frac{1-ne^{-ia}}{1-ne^{ia}}.$$

Taking logarithm of both sides, (considering the principal values only)

$$2ix=\log(1-ne^{-ia})-\log(1-ne^{ia})$$

$$=-ne^{-ia}-\tfrac{1}{2}n^2e^{-2ia}-\tfrac{1}{3}n^3e^{-3ia}-\cdots+ne^{ia}+\tfrac{1}{2}n^2e^{2ia}+\tfrac{1}{3}n^3e^{3ia}+\cdots$$

$$=n(e^{ia}-e^{-ia})+\tfrac{1}{2}n^2(e^{2ia}-e^{-2ia})+\tfrac{1}{3}n^3(e^{3ia}-e^{-3ia})+\cdots$$

$$=n.2i\sin a+\tfrac{1}{2}n^2.\,2i\sin 2a+\tfrac{1}{3}n^3.\,2i\sin 3a+\cdots\cdots$$

$$\therefore\quad x=n\sin a+\tfrac{1}{2}n^2\sin 2a+\tfrac{1}{3}n^3\sin 3a+\cdots\cdots$$

Ex. 5. *If B be an angle of a triangle ABC less than A, prove that*

$$B=\frac{b}{a}\sin C+\frac{1}{2}\frac{b^2}{a^2}\sin 2C+\frac{1}{3}\frac{b^3}{a^3}\sin 3C+\cdots\cdots$$

We have,

$$\sin B=\frac{b}{a}\sin A=\frac{b}{a}\sin(B+C).\qquad[\because A+B+C=\pi.]$$

Since this is of the form

$$\sin x=n\sin(x+a),\left[x=B,\ a=C,\ n=\frac{b}{a}\right]$$

hence, as in Ex. 4, we get

$$B=\frac{b}{a}\sin C+\frac{1}{2}\frac{b^2}{a^2}\sin 2C+\frac{1}{3}\frac{b^3}{a^3}\sin 3C+\cdot$$

Ex. 6. *If $\tan(\theta+\phi)\cos 2\beta=\tan\phi$, prove that*

$$\theta=\tan^2\beta\sin 2\phi+\tfrac{1}{2}\tan^4\beta\sin 4\phi+\tfrac{1}{3}\tan^6\beta\sin 6\phi+\cdots\cdots$$

Here, $\cos 2\beta=\dfrac{\tan\phi}{\tan(\theta+\phi)}$.

$$\therefore \quad \tan^2\beta = \frac{1-\cos 2\beta}{1+\cos 2\beta} = \frac{\tan(\theta+\phi)-\tan\phi}{\tan(\theta+\phi)+\tan\phi}$$

$$= \frac{\sin(\theta+\phi-\phi)}{\sin(\theta+\phi+\phi)}.$$

$$\therefore \quad \sin\theta = \tan^2\beta \sin(\theta+2\phi).$$

Since this is of the form of $\sin x = n\sin(x+a)$, hence, by **Ex. 4**, above, the required expansion follows.

Ex. 7. *If $\tan x = n\tan y$, find a series for x.* [*C. H. 1935, '42.*]

Here, $\dfrac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = n\cdot\dfrac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}}$.

$$\therefore \quad \frac{e^{2ix}-1}{e^{2ix}+1} = n\cdot\frac{e^{2iy}-1}{e^{2iy}+1}.$$

$$\therefore \quad e^{2ix} = \frac{(1+n)e^{2iy}+1-n}{(1-n)e^{2iy}+1+n}$$

$$= e^{2iy}\times\frac{1+ke^{-2iy}}{1+ke^{2iy}}, \text{ where } k = \frac{1-n}{1+n}.$$

$$\therefore \quad 2ix = 2iy + \log(1+ke^{-2iy}) - \log(1+ke^{2iy})$$

$$= 2iy - k(e^{2iy}-e^{-2iy}) + \frac{k^2}{2}(e^{4iy}-e^{-4iy}) - \ldots\ldots$$

$$\therefore \quad x = y - k\sin 2y + \frac{k^2}{2}\sin 4y - \frac{k^3}{3}\sin 6y + \ldots\ldots$$

Similarly, a series for y can be obtained.

Ex. 8. *Expand $e^{ax}\cos bx$ and $e^{ax}\sin bx$ in series of ascending powers of x.*

$$e^{ax}\cos bx = e^{ax}.\tfrac{1}{2}[e^{ibx}+e^{-ibx}] = \tfrac{1}{2}e^{(a+ib)x} + \tfrac{1}{2}e^{(a-ib)x}$$

$$= \tfrac{1}{2}[1+(a+ib)x+\frac{(a+ib)^2}{2!}x^2+\frac{(a+ib)^3}{3!}x^3+\cdots]$$

$$+\tfrac{1}{2}[1+(a-ib)x+\frac{(a-ib)^2}{2!}x^2+\frac{(a-ib)^3}{3!}x^3+\cdots].$$

The coefficient of $x^n = \frac{1}{2} \frac{(a+ib)^n+(a-ib)^n}{n\,!}$.

Let $a=r\cos\theta$; $b=r\sin\theta$; then $(a+ib)^n+(a-ib)^n$

$$=r^n(\cos\theta+i\sin\theta)^n+r^n(\cos\theta-i\sin\theta)^n=2r^n\cos n\theta$$

[by *De Moivre's Theorem.*]

$\therefore$ coefficient of $x^n=\dfrac{r^n\cos n\theta}{n\,!}$.

$$\therefore\quad e^{ax}\cos bx=1+r\cos\theta.x+\frac{r^2\cos 2\theta}{2\,!}.x^2+\cdot$$

where $r=\sqrt{a^2+b^2}$ and $\tan\theta=\dfrac{b}{a}$.

$$e^{ax}\sin bx=e^{ax}.\frac{1}{2i}[e^{ibx}-e^{-ibx}]=\frac{1}{2i}[e^{(a+ib)x}-e^{(a-ib)x}]$$

$$=\frac{1}{2i}[1+(a+ib)x+\frac{(a+ib)^2}{2\,!}x^2+\cdots$$

$$-\frac{1}{2i}[1+(a-ib)x+\frac{(a-ib)^2}{2\,!}x^2+\cdots].$$

As before, coefficient of $x^n=\dfrac{1}{2i}.\dfrac{(a+ib)^n-(a-ib)^n}{n\,!}$

$$=\frac{1}{2i}.\frac{2ir^n\sin n\theta}{n\,!}=\frac{r^n\sin n\theta}{n\,!}.$$

$$\therefore\quad e^{ax}\sin bx=r\sin\theta.x+\frac{r^2\sin 2\theta}{2\,!}x^2+\cdots\cdots$$

EXAMPLES X

1. If ϕ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, show that

$$\phi=\tfrac{1}{2}\pi-\cot\phi+\tfrac{1}{3}\cot^3\phi-\tfrac{1}{5}\cot^5\phi+\cdots$$

[*Put* $(\frac{1}{2}\pi-\phi)$ *for* θ *in Gregory's series*]

2. Show that

(i) $\pi = 2\sqrt{3}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right]$. [*C. P. 1943.*]

(ii) $\frac{\pi}{4} = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \cdots$

[*C. P. 1945, '47.*]

(iii) $\frac{\pi}{4} = \frac{17}{21} - \frac{713}{81 \times 343} + \cdots\cdots$

$$+ \frac{(-1)^{n+1}}{2n-1} \cdot \left\{\frac{2}{3^{2n-1}} + \frac{1}{7^{2n-1}}\right\} + \cdots\cdots$$

[*(iii) Break up every term of the series as in (ii).*]

(iv) $\frac{\pi}{12} = \left(1 - \frac{1}{3^{\frac{1}{2}}}\right) - \frac{1}{3}\left(1 - \frac{1}{3^{\frac{3}{2}}}\right) + \frac{1}{5}\left(1 - \frac{1}{3^{\frac{5}{2}}}\right) - \cdots$

[*C. P. 1949.*]

3. If x be $\lessdot (\sqrt{2} - 1)$, prove that

$$2(x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots)$$

$$= \frac{2x}{1-x^2} - \frac{1}{3}\cdot\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\cdot\left(\frac{2x}{1-x^2}\right)^5 - \cdots$$

4. If $0 < \theta < \frac{1}{2}\pi$, prove that

$$\tan^{-1}\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \cdots$$

5. If $\tan x < 1$, show that

$$\tan^2 x - \tfrac{1}{2}\tan^4 x + \tfrac{1}{3}\tan^6 x - \cdots$$

$$= \sin^2 x + \tfrac{1}{2}\sin^4 x + \tfrac{1}{3}\sin^6 x + \cdots$$

6. If $\theta < \frac{1}{4}\pi$, prove that

(i) $\log\sec\theta = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \cdots$

[*C. H. 1937.*]

(ii) $\frac{1}{2}\sin^2 2\theta = 2\tan^2\theta - 4\tan^4\theta + 6\tan^6\theta - \cdots$

7. (i) Expand $\log(\sec x + \tan x)$ in ascending powers of $\sin x$.

(ii) Expand $\log \cos x$ in ascending powers of $\tan x$, when x lies between $\pm \frac{1}{4}\pi$.

8. (i) Show that the coefficient of x^n in the expansion of $e^x \cos x$ in powers of x is $\dfrac{2^{\frac{1}{2}n}}{n\,!} \cos \dfrac{n\pi}{4}$. [*C. H. 1929.*]

(ii) Find the coefficient of x^n in the expansion of $e^{ax} \sin bx + e^{bx} \sin ax$ in powers of x. [*C. H. 1943.*]

9. Reduce $\tan^{-1}(\cos\theta + i \sin\theta)$ to the form $a + ib$ and hence show that

(i) $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \cdots = \pm \dfrac{\pi}{4}$.

[*C. H. 1931.*]

(ii) $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \cdots$

$$= \frac{1}{2} \log\left\{\pm \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}$$

the upper or lower sign being taken according as $\cos\theta$ is positive or negative.

10. If $\tan\theta = \dfrac{a \sin \alpha}{1 + a \cos \alpha}$, prove that

$$\theta = a \sin \alpha - \frac{a^2}{2} \sin 2\alpha + \frac{a^3}{3} \sin 3\alpha - \cdots$$ [*C. P. 1951.*]

[*This is only another form of Ex. 4 worked out.*]

11. Prove that

$$\frac{\tan^{-1}x}{x} + \frac{\tan^{-1}y}{y} + \frac{\tan^{-1}z}{z} = 3\left[1 - \frac{1}{7} + \frac{1}{13} - \frac{1}{19} + \frac{1}{25} - \cdots\right]$$

where x, y, z are the three cube roots of unity.

12. If θ be a positive acute angle, prove that

(i) $\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)=2(\sin\theta-\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta-\cdots)$.

(ii) $\log\cos\theta=-\log 2+\cos 2\theta-\frac{1}{2}\cos 4\theta+\frac{1}{3}\cos 6\theta-\cdots$

(iii) $$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}$$
$$=4[c\sin^2\theta-\tfrac{1}{2}c^2\sin^2 2\theta+\tfrac{1}{3}c^3\sin^3 3\theta-\cdots]$$
where $c=\dfrac{a-b}{a+b}$.

(iv) $\log(1-2x\cos\theta+x^2)$, where $-1<x<1$,
$$=-2(x\cos\theta+\tfrac{1}{2}x^2\cos 2\theta+\tfrac{1}{3}x^3\cos 3\theta+\cdots).$$
[*C. P. 1942.*]

13. Prove that
$$(\tan^{-1}x)^2=x^2-\tfrac{1}{2}(1+\tfrac{1}{3})x^4+\tfrac{1}{3}(1+\tfrac{1}{3}+\tfrac{1}{5})x^6-\cdots,$$
where x lies between ± 1. [*C. H. 1939.*]

14. Prove that in any triangle,
$$\log\frac{b}{a}=(\cos 2A-\cos 2B)+\frac{1}{2}(\cos 4A-\cos 4B)$$
$$+\frac{1}{3}(\cos 6A-\cos 6B)+\cdots$$

15. Given that $y=\log\tan\left[\dfrac{\pi}{4}+\dfrac{x}{2}\right]=x+c_3x^3+c_5x^5+\cdots$, show that $x=y-c_3y^3+c_5y^5-\cdots$ [*C. P. 1941 ; C. H. 1933.*]

16. (i) Prove that $\log(1+i\tan\theta)=\log\sec\theta+i\theta$ and hence deduce the expansion for θ in powers of $\tan\theta$.

(ii) Prove that $\log(1+ix)=\log\sqrt{1+x^2}+i\tan^{-1}x$ and hence deduce the expansion of $\tan^{-1}x$ in powers of x.

17. Find the sum to infinity of

(i) $1-\frac{1}{3.2^2}+\frac{1}{5.2^4}-\cdots$

(ii) $\frac{7}{1.3.5}+\frac{19}{5.7.9}+\frac{31}{9.11.13}+\cdots$

the numerators being in A. P.

$\left[\text{(ii) } u_n=\frac{1}{2}\left\{\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{2}{4n+1}\right\}.\right]$

18. Find the sum to infinity of

$$\tfrac{1}{3}x^3+\tfrac{1}{7}x^7+\tfrac{1}{11}x^{11}+\cdots$$

19. When both θ and $\tan^{-1}(\sec\theta)$ lie between 0 and $\frac{1}{2}\pi$, prove that

$$\tan^{-1}(\sec\theta)=\frac{\pi}{4}+\tan^2\frac{\theta}{2}-\frac{1}{3}\tan^6\frac{\theta}{2}+\frac{1}{5}\tan^{10}\frac{\theta}{2}-\cdots$$

20. In any triangle when $a<c$, show that

$$\frac{\cos nA}{b^n}=\frac{1}{c^n}\left\{1+\frac{na}{c}\cos B+\frac{n(n+1)}{1.2}\cdot\frac{a^2}{c^2}\cos 2B+\cdots\right\}$$

$$\frac{\sin nA}{b^n}=\frac{n}{c^n}\left\{\frac{a}{c}\sin B+\frac{n+1}{1.2}\cdot\frac{a^2}{c^2}\sin 2B+\cdots\right\}.$$

21. In any triangle where $a>b$, show that

$$\log c=\log a-\frac{b}{a}\cos C-\frac{1}{2}\frac{b^2}{a^2}\cos 2C-\frac{1}{3}\frac{b^3}{a^3}\cos 3C-\cdots$$

[*C. P. 1942.*]

$\left[c^2=a^2+b^2-2ab\cos C=a^2\left(1-\frac{b}{a}e^{iC}\right)\left(1-\frac{b}{a}e^{-iC}\right)\right]$

22. Expand in a series of sines and cosines of multiples of x

(i) $\log(a\cos x+b\sin x)$.

(ii) $\log\cos\left(x+\frac{\pi}{4}\right)$.

23. Show that if x lies between ± 1,

$$\log(\tan^{-1}x) - \log x = -\tfrac{1}{3}x^2 + \tfrac{13}{90}x^4 - \cdots$$

24. If $\tan\dfrac{\phi}{2} = 2\tan\dfrac{\theta}{2}$, prove that

$$\frac{1}{2}(\phi - \theta) = \frac{1}{1.3}\sin\theta + \frac{1}{2.3^2}\sin 2\theta + \frac{1}{3.3^3}\sin 3\theta + \cdots$$

25. If the roots of the equation $ax^2 + bx + c = 0$ be imaginary, show that the coefficient of x^n in the development of $(ax^2 + bx + c)^{-1}$ in powers of x is

$$\{a^{\frac{1}{2}n}\sin(n+1)\theta\} \div \{c^{\frac{1}{2}n+1}\sin\theta\},$$

where θ is given by $b\sec\theta + 2\sqrt{ac} = 0$. [*C. H. 1931, '39.*]

26. If $\tan x = \dfrac{n\sin a}{1 - n\cos a}$, $n < 1$, prove that

$$x = n\sin a + \tfrac{1}{2}n^2\sin 2a + \tfrac{1}{3}n^3\sin 3a + \cdots$$

[*C. P. 1939.*]

27. If $\tan\theta = x + \tan a$, prove that

$$\theta = a + x\cos^2 a - \tfrac{1}{2}x^2\cos a\sin 2a - \tfrac{1}{3}x^3\cos^3 a\cos 3a$$
$$+ \tfrac{1}{4}x^4\cos^4 a\sin 4a + \cdots$$

[*C. H. 1940.*]

28. If $\cot y = \cot x + \operatorname{cosec} a \operatorname{cosec} x$, show that

$$y = \sin x\sin a - \tfrac{1}{2}\sin 2x\sin^2 a + \tfrac{1}{3}\sin 3x\sin^3 a - \cdots$$

[*C. P. 1938.*]

ANSWERS

7. (i) $\sin x + \frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + \cdots$

(ii) $-\frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x - \frac{1}{6}\tan^6 x + \cdots$

8. (ii) $\dfrac{(a^2+b^2)^{\frac{1}{2}n}}{n\,!}\left\{\sin\left(n\tan^{-1}\dfrac{b}{a}\right) + \sin\left(n\tan^{-1}\dfrac{a}{b}\right)\right\}$

17. (i) $2\tan^{-1}\frac{1}{2}$. (ii) $1 - \frac{1}{8}\pi$. **18.** $\frac{1}{4}\log\dfrac{1+x}{1-x}$

22. (i) $\log\frac{1}{2}\sqrt{(a^2+b^2)} + \cos 2(x-a) - \frac{1}{2}\cos 4(x-a)$

$+ \frac{1}{3}\cos 6(x-a) - \cdots\cdots$, where $\tan a = \dfrac{b}{a}$.

(ii) $-\log 2 - \sin 2x + \frac{1}{2}\cos 4x + \frac{1}{3}\sin 6x - \frac{1}{4}\cos 8x - \frac{1}{5}\sin 10x + \cdots$

CHAPTER XI

SUMMATION OF TRIGONOMETRICAL SERIES

75. Method of Difference.

When the rth term of a trigonometrical series can be expressed as the difference of two quantities, one of which is the same function of r as the other is of $(r+1)$, the sum of the series may be readily found as illustrated in the Examples 1 and 2.

Ex. 1. *Find the sum of the series*

(*i*) $\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}2^2\theta+\cdots\cdots+\operatorname{cosec}2^{n-1}\theta.$

[*C. P. 1936.*]

(*ii*) $\dfrac{\sin x}{\sin 2x\sin 3x}+\dfrac{\sin x}{\sin 3x\sin 4x}+\dfrac{\sin x}{\sin 4x\sin 5x}+\cdots$ *to n terms.*

(i) We have, $\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}=\dfrac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\sin\theta}$

$$=\frac{\sin(\theta-\frac{1}{2}\theta)}{\sin\frac{1}{2}\theta\sin\theta}=\frac{\sin\theta\cos\frac{1}{2}\theta-\cos\theta\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\sin\theta}$$

$$=\cot\tfrac{1}{2}\theta-\cot\theta.$$

Thus, $\qquad \operatorname{cosec}\theta=\cot\frac{1}{2}\theta-\cot\theta.$

Similarly, $\qquad \operatorname{cosec}2\theta=\cot\theta-\cot 2\theta$

$$\operatorname{cosec}2^2\theta=\cot 2\theta-\cot 2^2\theta$$

$$\operatorname{cosec}2^{n-1}\theta=\cot 2^{n-2}\theta-\cot 2^{n-1}\theta$$

$\therefore$ by addition, the required sum

$$=\cot\tfrac{1}{2}\theta-\cot 2^{n-1}\theta.$$

(ii) Here, rth term $= \dfrac{\qquad}{\sin(r+1)x \sin(r+2)x}$

$$= \frac{\sin\{(r+2)-(r+1)\}x}{\sin(r+1)x \sin(r+2)x}$$

$$= \frac{\sin(r+2)x\cos(r+1)x - \cos(r+2)x\sin(r+1)x}{\sin(r+1)x\sin(r+2)x}$$

$$= \cot(r+1)x - \cot(r+2)x.$$

Putting $r=1, 2, 3, \ldots n$, and adding, the sum of the required series would be found to be

$$\cot 2x - \cot(n+2)x.$$

Ex. 2. *Find the sum of the series*

$$\tan^{-1}\frac{x}{1+1.2x^2} + \tan^{-1}\frac{x}{1+2.3x^2} + \cdots\cdots + \tan^{-1}\frac{x}{1+n(n+1)x^2}.$$

Here, $\quad r$th term $= \tan^{-1}\dfrac{x}{1+r(r+1)x^2} = \tan^{-1}\dfrac{(r+1)x - rx}{1+(r+1)x.rx}$

$$= \tan^{-1}(r+1)x - \tan^{-1}rx.$$

$\therefore$ putting $r = 1, 2, 3, \ldots$ we have,

$$\tan^{-1}\frac{x}{1+1.2x^2} = \tan^{-1}2x - \tan^{-1}x$$

$$\tan^{-1}\frac{x}{1+2.3x^2} = \tan^{-1}3x - \tan^{-1}2x$$

....

$$\tan^{-1}\frac{x}{1+n(n+1)x^2} = \tan^{-1}(n+1)x - \tan^{-1}nx.$$

$\therefore$ by addition, the required sum

$$= \tan^{-1}(n+1)x - \tan^{-1}x.$$

76. Sometimes the rth term of a series, *when multiplied by a factor*, can be expressed as the difference of two quantities one of which is the same function of r as the other is of $(r+1)$. It is illustrated in the following two cases.

(i) Sum of sines of n angles in A. P.

Let the angles in A. P. be α, $\alpha+\beta$, $\alpha+2\beta,\ldots\{\alpha+(n-1)\beta\}$ the first term being α, and the common difference, β.

Let S denote the sum of the series

$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots\cdots+\sin\{\alpha+(n-1)\beta\}$.

Multiplying each term of the above series by

2 sin (half difference) i.e., by $2\sin\frac{1}{2}\beta$, we have,

$$2\sin\alpha.\sin\tfrac{1}{2}\beta=\cos(\alpha-\tfrac{1}{2}\beta)-\cos(\alpha+\tfrac{1}{2}\beta)$$
$$2\sin(\alpha+\beta).\sin\tfrac{1}{2}\beta=\cos(\alpha+\tfrac{1}{2}\beta)-\cos(\alpha+\tfrac{3}{2}\beta)$$
$$2\sin(\alpha+2\beta).\sin\tfrac{1}{2}\beta=\cos(\alpha+\tfrac{3}{2}\beta)-\cos(\alpha+\tfrac{5}{2}\beta)$$
$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

$$2\sin\{\alpha+(n-1)\beta\}.\sin\tfrac{1}{2}\beta$$
$$=\cos\left(\alpha+\frac{2n-3}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right).$$

Adding vertically, we have

$$2\sin\tfrac{1}{2}\beta.S=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$
$$=2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}.$$

$$\therefore\ S=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right).$$

Cor. Putting $\beta=\alpha$, we get

$$\sin\alpha+\sin2\alpha+\sin3\alpha+\cdots\cdots+\sin n\alpha$$

$$=\frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.\sin\frac{n+1}{2}\alpha.$$

(ii) Sum of cosines of n angles in A. P.

As before, let S denote the sum of the series

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \cdots + \cos \{\alpha + (n-1)\beta\}.$$

Multiplying each term of the above series by

2 *sin* (*half difference*), we have

$$2 \cos \alpha . \sin \tfrac{1}{2}\beta = \sin (\alpha + \tfrac{1}{2}\beta) - \sin (\alpha - \tfrac{1}{2}\beta)$$

$$2 \cos (\alpha + \beta) . \sin \tfrac{1}{2}\beta = \sin (\alpha + \tfrac{3}{2}\beta) - \sin (\alpha + \tfrac{1}{2}\beta)$$

$$2 \cos (\alpha + 2\beta) . \sin \tfrac{1}{2}\beta = \sin (\alpha + \tfrac{5}{2}\beta) - \sin (\alpha + \tfrac{3}{2}\beta)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$2 \cos \{\alpha + (n-1)\beta\} . \sin \tfrac{1}{2}\beta$$

$$= \sin \left(\alpha + \frac{2n-1}{2}\beta\right) - \sin \left(\alpha + \frac{2n-3}{2}\beta\right).$$

Adding vertically, we have,

$$2 \sin \tfrac{1}{2}\beta . S = \sin \left(\alpha + \frac{2n-1}{2}\beta\right) - \sin \left(\alpha - \frac{\beta}{2}\right)$$

$$= 2 \cos \left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2};$$

$$\therefore \quad S = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cdot \cos \left(\alpha + \frac{n-1}{2}\beta\right).$$

Cor. Putting $\beta = \alpha$, we get

$$\cos \alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos n\alpha = \frac{\sin \frac{n\alpha}{2}}{\sin \frac{\alpha}{2}} \cos \frac{n+1}{2}\alpha.$$

Note. The sum of the cosine series may also be deduced from that of the sine series by writing $\alpha + \frac{\pi}{2}$ for α.

77. As an aid to memory, the two formulæ of the last article may be expressed verbally as follows :

Since, $\alpha+\frac{n-1}{2}\beta=\frac{\alpha+\alpha+(n-1)\beta}{2}$,

$\therefore$ **Sum of sines of n angles in A. P.**

$$=\frac{\sin\frac{n.\ diff.}{2}}{\sin\frac{diff.}{2}}\sin\frac{first\ angle+last\ angle}{2}.$$

Sum of cosines of n angles in A. P.

$$=\frac{\sin\frac{n.\ diff.}{2}}{\sin\frac{diff.}{2}}\cos\frac{first\ angle+last\ angle}{2}.$$

Note. From the above formulæ, it is clear that if $\sin\frac{n\beta}{2}=0$, then the sum of the sine series as also the sum of the cosine series is zero.

Now, if $\sin\frac{n\beta}{2}=0$, then $\frac{n\beta}{2}=k\pi$, or, $\beta=\frac{2k\pi}{n}$, where k is an integer.

Thus, *the sum of the sines and the sum of the cosines of n angles in A. P. are each equal to zero when the common difference of the angles is an even multiple of* $\frac{\pi}{n}$.

Ex. 8. *Find the sum of n terms of the series*

$\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\cdots$ [*C. P. 1928, '39.*]

Since, $\sin(\pi+\theta)=-\sin\theta$; $\sin(2\pi+\theta)=\sin\theta$; etc.

$\therefore$ the series is equal to

$$\sin\alpha+\sin(\pi+\alpha+\beta)+\sin(2\pi+\alpha+2\beta)+\cdots\cdots$$

i.e., equal to a series in which the common difference of the angles is $\beta+\pi$ and the last angle is $\alpha+(n-1)(\beta+\pi)$.

$$\therefore\ S=\frac{\sin\frac{n(\beta+\pi)}{2}}{\sin\frac{\beta+\pi}{2}}\sin\left\{\alpha+\frac{(n-1)(\beta+\pi)}{2}\right\}.$$

Ex. 4. *Find the sum of the series*

$$\sin^2\theta+\sin^2 2\theta+\sin^2 3\theta+\cdots+\sin^2 n\theta. \qquad [\text{ C. P. 1924. }]$$

Since, $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$; $\sin^2 2\theta=\frac{1}{2}(1-\cos 4\theta)$; etc.

$\therefore$ the given series $=\frac{1}{2}(1-\cos 2\theta)+\frac{1}{2}(1-\cos 4\theta)+\cdots+\frac{1}{2}(1-\cos 2n\theta)$

$$=\frac{n}{2}-\frac{1}{2}(\cos 2\theta+\cos 4\theta+\cdots+\cos 2n\theta)$$

$$=\frac{n}{2}-\frac{1}{2}\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta \text{ by } Art.\ 76.$$

78. C+iS method of summation.

When the sum of a finite or infinite series of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\cdots\cdots \qquad \cdots \quad (1)$$

is known, we can obtain the sums of the two series of the form

$$a_0+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta+\cdots\cdots \qquad (2)$$

and $$a_1\sin\theta+a_2\sin 2\theta+a_3\sin 3\theta+\cdots\cdots\cdots\cdots \qquad (3)$$

Let $f(x)$ denote the sum of the series (1), and C denote the sum of the series (2) and S denote the sum of the series (3).

Now, $C+iS$

$$=a_0+a_1(\cos\theta+i\sin\theta)+a_2(\cos 2\theta+i\sin 2\theta)+\cdots$$
$$=a_0+a_1e^{i\theta}+a_2e^{2i\theta}+\cdots$$
$$=f(e^{i\theta}). \qquad \cdots \qquad \cdots \qquad \cdots \quad (4)$$

Similarly. $\qquad C-iS=f(e^{-i\theta}). \qquad \cdots \quad (5)$

Adding (4) and (5), $\quad C=\frac{1}{2}[f(e^{i\theta})+f(e^{-i\theta})].$

Subtracting (5) from (4), $S=\frac{1}{2i}[f(e^{i\theta})-f(e^{-i\theta})].$

Cor. The sum of the two series of the forms

$$a_0\cos\alpha+a_1\cos(\alpha+\theta)+a_2\cos(\alpha+2\theta)+\cdots \quad (6)$$
$$a_0\sin\alpha+a_1\sin(\alpha+\theta)+a_2\sin(\alpha+2\theta)+\cdots \quad (7)$$

can also be obtained by the above method.

Note 1. If θ be *real*, then C and S can be obtained by expressing $f(e^{i\theta})$ in the form $A+iB$ and then equating real and imaginary parts from both sides of the relation (4) ; then $C=A$ and $S=B$.

Note 2. The cosine and sine series of Art. 76 are particular cases of series (6) and (7), when a_0, a_1, a_2 etc. are all equal to 1, and $\theta=\beta$. Hence, *the cosine and sine series of Art. 76 can also be summed by this method.* In this case,

$$C+iS=e^{i\alpha}+e^{i(\alpha+\beta)}+e^{i(\alpha+2\beta)}+\cdots\cdots \text{ to } n \text{ terms}$$

which is a series in G. P. the first term being $e^{i\alpha}$ and the common ratio $e^{i\beta}$.

Ex. 5. *Sum the series*

$$\cos\theta-\tfrac{1}{2}\cos 2\theta+\tfrac{1}{3}\cos 3\theta-\cdots\cdots \; ad\ inf\ (-\pi<\theta<\pi).$$

[*C. P. 1939, '45, '48.*]

Let $\quad C=\cos\theta-\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta-\cdots\cdots$

$\quad S=\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\cdots$

$$\therefore\quad C+iS=e^{i\theta}-\tfrac{1}{2}e^{2i\theta}+\tfrac{1}{3}e^{3i\theta}-\cdots\cdots$$

$$=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\cdots\cdots\ [\text{putting } x=e^{i\theta}]$$

$$=\log(1+x)=\log(1+e^{i\theta}). \qquad \cdots \quad (1)$$

Similarly, $C-iS=\log(1+e^{-i\theta}). \qquad \cdots \quad (2)$

Adding (1) and (2), $2C=\log(1+e^{i\theta})+\log(1+e^{-i\theta})$

$$=\log[(1+e^{i\theta})(1+e^{-i\theta})]$$

$$=\log\{1+1+e^{i\theta}+e^{-i\theta}\}$$

$$=\log(2+2\cos\theta)=\log(2.2\cos^2\tfrac{1}{2}\theta)$$

$$=\log(2\cos\tfrac{1}{2}\theta)^2=2\log(2\cos\tfrac{1}{2}\theta).$$

ıired sum) $=\log(2\cos\frac{1}{2}\theta)$.

Note 1. The above series can also be summed in the following way. Writing down the exponential values of $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$ etc. and re-arranging the terms, the series may be written as

$$\tfrac{1}{2}(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\cdots)+\tfrac{1}{2}(y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\cdots\cdots),$$

$$[\text{where } x=e^{i\theta} \text{ and } y=e^{-i\theta}]$$

or, $$=\tfrac{1}{2}\log(1+x)+\tfrac{1}{2}\log(1+y)$$

$$=\tfrac{1}{2}\log\left(1+e^{i\theta}\right)+\tfrac{1}{2}\log\left(1+e^{-i\theta}\right)=\text{etc.}$$

Note 2. Where θ is *real*, the sum of the series can also be obtained thus :

$$C+iS=\log\left(1+e^{i\theta}\right)=\log(1+\cos\theta+i\sin\theta)$$

$$=\log(2\cos^2\tfrac{1}{2}\theta+2i\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)$$

$$=\log\{2\cos\tfrac{1}{2}\theta(\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta)\}=\log\left\{2\cos\tfrac{1}{2}\theta . e^{\frac{1}{2}i\theta}\right\}$$

$$=\log(2\cos\tfrac{1}{2}\theta)+\log e^{\frac{1}{2}i\theta}=\log(2\cos\tfrac{1}{2}\theta)+\frac{i\theta}{2}.$$

Now equate the real parts.

Ex. 6. *Sum the series*

$$1+x\cos\theta+x^2\cos 2\theta+\cdots\cdots+x^{n-1}\cos(n-1)\theta.$$

Let $$C=1+x\cos\theta+x^2\cos 2\theta+\cdots\cdots+x^{n-1}\cos(n-1)\theta\,;$$

$$S=x\sin\theta+x^2\sin 2\theta+\cdots\cdots+x^{n-1}\sin(n-1)\theta.$$

$\therefore\quad C+iS=1+xe^{i\theta}+x^2e^{2i\theta}+\cdots\cdots+x^{n-1}e^{(n-1)i\theta}.$

This is a series in G. P,

$\therefore\quad$ its sum $=\dfrac{1-x^n e^{in\theta}}{1-xe^{i\theta}}.$

Multiplying the numerator and the denominator of this sum by $1-xe^{-i\theta}$ we have,

$$C+iS=\frac{\left(1-x^ne^{in\theta}\right)\left(1-xe^{-i\theta}\right)}{\left(1-xe^{i\theta}\right)\left(1-xe^{-i\theta}\right)}$$

$$=\frac{1-xe^{-i\theta}-x^ne^{in\theta}+x^{n+1}e^{i(n-1)\theta}}{1-2x\cos\theta+x^2}.$$

Now, the denominator is real and the numerator when simplified will be found to consist of a real part and an imaginary part. Therefore, equating the real parts, we get

$$C = \frac{1 - x\cos\theta - x^n \cos n\theta + x^{n+1}\cos(n-1)\theta}{1 - 2x\cos\theta + x^2}.$$

Ex. 7. *Sum the series*

$$\cos \alpha + 2\cos(\alpha+\beta) + 3\cos(\alpha+2\beta) + \cdots + n\cos\{\alpha+(n-1)\beta\}.$$

[*C. H. 1914.*]

Let u_r, denote the rth term and S denote the sum of the given series.

Now, $2\cos\beta . u_r = 2\cos\beta . r\cos\{\alpha+(r-1)\beta\}$

$$= r\,[\cos(\alpha+r\beta) + \cos\{\alpha+(r-2)\beta\}].$$

$\therefore$ putting $r = 1, 2, 3, \ldots n$ and adding together, we get $2\cos\beta . S$.

Now, subtract $2\cos\beta . S$ from $2S$; then

$$2S(1-\cos\beta) = (n+1)\cos\{\alpha+(n-1)\beta\} - \cos(\alpha-\beta) - n\cos(\alpha+n\beta).$$

Dividing by $2(1-\cos\beta)$, S, the sum of the required series would be obtained.

Note. Similarly the sum of the series

$$\sin\alpha + 2\sin(\alpha+\beta) + 3\sin(\alpha+2\beta) + \cdots + n\sin\{\alpha+(n-1)\beta\}$$

would be obtained. [*C. H. 1922.*]

EXAMPLES XI

1. Sum to n terms the following series :

(i) $\sin\alpha + \sin\left(\alpha + \frac{\pi}{n}\right) + \sin\left(\alpha + \frac{2\pi}{n}\right) + \cdots\cdots$

[*C. P. 1932.*]

(ii) $\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cos\left(\alpha + \frac{4\pi}{n}\right) + \cdots\cdots$

(iii) $\sin\alpha - \sin 2\alpha + \sin 3\alpha - \cdots\cdots$ [*C. P. 1929.*]

(iv) $\cos^2\alpha + \cos^2 3\alpha + \cos^2 5\alpha + \cdots$ [*C. P. 1930.*]

(v) $\sin^3\alpha + \sin^3(\alpha+\beta) + \sin^3(\alpha+2\beta) + \cdots$

[*C. P. 1931, '37.*]

(vi) $\sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \cdots$

(vii) $\sin^4\alpha + \sin^4 2\alpha + \sin^4 3\alpha + \cdots\cdots$

(viii) $\sin 3\theta \sin\theta + \sin 6\theta \sin 2\theta + \sin 12\theta \sin 4\theta + \cdots$

(ix) $\sin^3\alpha + \frac{1}{3}\sin^3 3\alpha + \frac{1}{3^2}\sin^3 3^2\alpha + \frac{1}{3^3}\sin^3 3^3\alpha + \cdots$

(x) $\sin(p+1)\alpha \cos\alpha + \sin(p+2)\alpha \cos 2\alpha + \cdots$

[*C. P. 1935.*]

(xi) $\sin\alpha \sin 3\alpha + \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2} + \sin\frac{\alpha}{2^2}\sin\frac{3\alpha}{2^2} + \cdots$

[*C. P. 1936.*]

(xii) $\cos\theta\cos 2\theta + \cos 3\theta \cos 4\theta + \cos 5\theta \cos 6\theta + \cdots$

(xiii) $\sin x \cos x - \sin 2x \cos 2x + \sin 3x \cos 3x - \cdots$

(xiv) $\sin^2\theta \sin 2\theta + \sin^2 2\theta \sin 3\theta + \sin^2 3\theta \sin 4\theta + \cdots$

[*C. P. 1941.*]

(xv) $\sqrt{1+\sin\alpha} + \sqrt{1+\sin 2\alpha} + \sqrt{1+\sin 3\alpha} + \cdots$

[*C. H. 1941.*]

2. Prove that :

(i) $$\frac{\sin\theta + \sin 3\theta + \sin 5\theta + \cdots \text{ to } n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \cdots \text{ to } n \text{ terms}} = \tan n\theta.$$

(ii) $$\sin^2\alpha + \sin^2\left(\alpha + \frac{2\pi}{n}\right) + \sin^2\left(\alpha + \frac{4\pi}{n}\right) + \cdots \text{ to } n \text{ terms}$$

$$= \tfrac{1}{2}n, \text{ except when } n = 1, \text{ or, } 2.$$

3. Find the sums of the series

(i) $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \cdots\cdots$ to n terms,

and (ii) $\cos\alpha + \cos 3\alpha + \cos 5\alpha + \cdots\cdots$ to n terms.

Hence, deduce respectively the sums of the series

(iii) $1 + 2 + 3 + \cdots\cdots$ to n terms. [*C. P. 1951.*]

and (iv) $1^2 + 3^2 + 5^2 + \cdots\cdots$ to n terms. [*C. P. 1942.*]

4. (i) Sum to n terms the series

$$\sin\alpha\sin 2\alpha + \sin 2\alpha\sin 3\alpha + \sin 3\alpha\sin 4\alpha + \cdots$$

and (ii) hence deduce the sum of the series

$$1.2 + 2.3 + 3.4 + \cdots \text{ to } n \text{ terms.}$$ [*C. P. 1940.*]

5. If β be the exterior angle of a regular polygon of n sides, show that,

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots$ to n terms $= 0$.

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots$ to n terms $= 0$.

[*Here,* $\beta = \frac{2\pi}{n}$. *Now, use note of Art. 77.*]

6. Show that the sum of the series

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \cdots \text{ to } n \text{ terms,}$$

where $n\beta = 2\pi$ and x is one of the nth roots of unity, vanishes with two exceptions, and find the sum in these two cases.

7. Sum to infinity the following series :

(i) $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \cdots\cdots$ [*C. P. 1940.*]

(ii) $\sin\alpha + \dfrac{1}{2}\sin 2\alpha + \dfrac{1}{2^2}\sin 3\alpha + \cdots\cdots$

(iii) $\cos\theta + \dfrac{\operatorname{cosec}\theta}{1!}\cdot\cos 2\theta + \dfrac{\operatorname{cosec}^2\theta}{2!}\cdot\cos 3\theta + \cdots$

(iv) $2x\cos\theta + \dfrac{3x^2}{2}\cos 2\theta + \dfrac{4x^3}{3}\cos 3\theta + \cdots\cdots$

(v) $1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\cdots,$
where θ lies between $\pm\pi$. [*C. H. 1931, '39, '41.*]

[*Here,* $C+iS=(1+e^{i\theta})^{-\frac{1}{2}}$]

(vi) $\sin\alpha+x\sin(\alpha+\beta)+\frac{x^2}{2!}\sin(\alpha+2\beta)+\cdots\cdots$

[*C. H. 1940.*]

(vii) $\sin\theta.\ \sin\theta-\frac{1}{2}\sin 2\theta.\ \sin^2\theta+\frac{1}{3}\sin 3\theta\ \sin^3\theta-\cdots$

(viii) $2\cos\theta+\frac{3}{2}\cos^2\theta+\frac{4}{3}\cos^3\theta+\frac{5}{4}\cos^4\theta+\cdots\cdots$

(ix) $2\tan\theta-\frac{4}{3}\tan^3\theta+\frac{6}{5}\tan^5\theta-\frac{8}{7}\tan^7\theta+\cdots\cdots$
$(-\frac{1}{4}\pi<\theta<\frac{1}{4}\pi)$ [*C. P. 1942.*]

(x) $\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\cdots\cdots\ (-\pi<\theta<\pi)$
[*C. P. 1941.*]

(xi) $\cos\theta\cos\theta+\cos^2\theta\cos 2\theta+\cos^3\theta\cos 3\theta+\cdots(\theta\neq n\pi)$

(xii) $1+\cos x+\frac{\cos 2x}{2!}+\frac{\cos 3x}{3!}+\cdots$

(xiii) $c\cos a-\frac{1}{2}c^2\cos 2a+\frac{1}{3}c^3\cos 3a-\cdots\ (0<c<1)$

(xiv) $m\sin^2 a-\frac{1}{2}m^2\sin^2 2a+\frac{1}{3}m^3\sin^2 3a-\cdots\ (0<m<1)$
[*C. H. 1936.*]

(xv) $\frac{\sin x}{\pi}-\frac{\sin 2x}{\pi^2}+\frac{\sin 3x}{\pi^3}-\cdots\cdots$

(xvi) $\cos x\sin x+\frac{1}{2}\cos^2 x\sin 2x+\frac{1}{3}\cos^3 x\sin 3x+\cdots$
$(0<x<\pi)$

8. Sum to n terms the series :

(i) $\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\cdots\cdots$

[$\tan 2^r\alpha=\cot 2^r\alpha-2\cot 2^{r+1}\alpha$]

(ii) $\frac{1}{\sin\theta\sin 2\theta}+\frac{1}{\sin 2\theta\sin 3\theta}+\frac{1}{\sin 3\theta\sin 4\theta}\cdot$

(iii) $\dfrac{1}{\cos a+\cos 3a}+\dfrac{1}{\cos a+\cos 5a}+\dfrac{1}{\cos a+\cos 7a}+\cdots$

[*C. H. 1935.*]

(iv) $\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}+\tan^{-1}\dfrac{1}{13}+\cdots+\tan^{-1}\dfrac{1}{1+n(n+1)}$.

(v) $\tan^{-1}\dfrac{1}{2.1^2}+\tan^{-1}\dfrac{1}{2.2^2}+\tan^{-1}\dfrac{1}{2.3^2}+\cdots$

[*C. H. 1937.*]

[$\tan^{-1}\dfrac{1}{2.r^2}=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$]

(vi) $\sqrt{1+\sin 2a}+\sqrt{1+\sin 2(a+\beta)}$

$+\sqrt{1+\sin 2(a+2\beta)}+\cdots$ [*C. H. 1926.*]

[$\sin\theta+\cos\theta=\sqrt{2}\sin(\theta+\frac{1}{4}\pi)$ *and* $1+\sin 2\theta=(\sin\theta+\cos\theta)^2$]

(vii) $\tan^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}$

$+\tan^{-1}\dfrac{1}{1+3+3^2}+\cdots$

(viii) $\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.3^2)+\cdots$

[*C. H. 1948.*]

[$\cot^{-1}(2r^2)=\tan^{-1}\dfrac{1}{2.r^2}$; *compare Ex. 8(v)*]

(ix) $\cos\theta+2\cos 2\theta+3\cos 3\theta+\cdots+n\cos n\theta$.

[*Multiply the series by* $2(1-\cos\theta)$; *cf. Ex. 7, worked out.*]

(x) $\cos x\cos 2x\cos 3x+\cos 2x\cos 3x\cos 4x+\cdots$

(xi) $\dfrac{1}{2}\sec\theta+\dfrac{1}{2^2}\sec\theta\sec 2\theta+\dfrac{1}{2^3}\sec\theta\sec 2\theta\sec 2^2\theta+\cdots$

[*C. H. 1934.*]

[*1st term* $=\sin\theta(\cot\theta-\cot 2\theta)$]

9. Prove that

(i) $\log\cos\theta + \log\cos\frac{1}{2}\theta + \log\cos\frac{1}{2^2}\theta + \cdots$ to inf.

$$= \log\frac{\sin 2\theta}{2\theta}.$$

$$\left[\textit{Use } Lt \cos\frac{1}{2}\theta.\cos\frac{1}{2^2}\theta.\cos\frac{1}{2^3}\theta\cdots = \frac{\sin\theta}{\theta}. \right]$$

(ii) $\cos\frac{1}{3}\pi + \frac{1}{2}\cos\frac{2}{3}\pi + \frac{1}{3}\cos\frac{3}{3}\pi + \cdots\cdots$ to inf. $= 0$.

[$C + iS = -\log(1 - e^{i\theta})$, *where* $\theta = \frac{1}{3}\pi$]

(iii) $2(\cos^2 x - \frac{1}{2}\sin^2 2x + \frac{1}{3}\cos^2 3x - \frac{1}{4}\sin^2 4x + \cdots$ to inf.)

$= \log(\operatorname{cosec} x)$, if $x \neq n\pi$.

(iv) $\dfrac{c\sin B}{a+b} + \dfrac{1}{2}\dfrac{c^2\sin 2B}{(a+b)^2} + \dfrac{1}{3}\dfrac{c^3\sin 3B}{(a+b)^3} + \cdots$ to inf. $= \frac{1}{2}C$,

where a, b, c, B, C denote the sides and angles of the triangle ABC.

10. Sum the series

$$\tan x + \frac{1}{2}\tan\frac{1}{2}x + \frac{1}{2^2}\tan\frac{1}{2^2}x + \cdots + \frac{1}{2^{n-1}}\tan\frac{1}{2^{n-1}}x,$$

and find its limiting value when n is increased indefinitely.

[*C. H. 1932 ; C. P. 1933, '44.*]

[$\tan x = \cot x - 2\cot 2x$.]

11. Sum the series

$\tan x\tan 2x + \tan 2x\tan 3x + \cdots\cdots + \tan nx\tan(n+1)x$

and hence deduce the sum of the series

$$1.2 + 2.3 + \cdots\cdots + n(n+1).$$

12. Sum to n terms the series

$$1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta + \cdots$$

If this sum be denoted by s_n, prove that

$$s_1 + s_2 + \cdots\cdots + s_n = \sin^2\tfrac{1}{2}n\theta\operatorname{cosec}^2\tfrac{1}{2}\theta.$$

[*C. H. 1928.*]

13. If $P \equiv \sin\theta + \sin 3\theta + \sin 5\theta + \cdots\cdots$ to n terms,

$$Q \equiv \cos\theta + \cos 3\theta + \cos 5\theta + \cdots\cdots \text{ to } n \text{ terms,}$$

and if a right-angled triangle be constructed such that the sides which include the right angle are of lengths P and Q respectively, prove that the length R of the hypotenuse can be had as a rational function of $\cos\theta$ of degree $n-1$.

Also obtain the actual expression for R when $n=5$.

[*C. H. 1930.*]

14. If s_n be the sum to n terms of the series

$$\sin x + \sin 2x + \sin 3x + \cdots\cdots,$$

prove that $\displaystyle \underset{n\to\infty}{Lt} \frac{s_1 + s_2 + \cdots + s_n}{n} = \tfrac{1}{2}\cot\tfrac{1}{2}x.$

15. If $c = \cos^2\theta - \frac{1}{3}\cos^3\theta\cos 3\theta + \frac{1}{5}\cos^5\theta\cos 5\theta - \cdots\cdots$, show that $\tan 2c = 2\cot^2\theta$. [*C. H. 1934, '40.*]

16. A regular polygon of n sides is inscribed in a circle of radius a; prove that

(i) the sum of the lengths of the perpendiculars drawn from the angular points to any diameter is zero;

(ii) the sum of the lengths of the lines joining any one vertex to each of the other vertices is $2a\cot\dfrac{\pi}{2n}$.

ANSWERS

1. (i) $\dfrac{\sin\left(\alpha + \frac{n-1}{2}\cdot\frac{\pi}{n}\right)}{\sin\frac{\pi}{2n}}$. (ii) 0.

(iii) $\dfrac{\sin\left\{\alpha + \frac{n-1}{2}(\alpha+\pi)\right\}\sin\frac{n(\alpha+\pi)}{2}}{\sin\frac{\alpha+\pi}{2}}$. (iv) $\dfrac{1}{2}\left(n + \dfrac{\sin 4n\alpha}{2\sin 2\alpha}\right)$.

(v) $\dfrac{1}{4}\left\{3\dfrac{\sin\left(\alpha+\dfrac{n-1}{2}\beta\right)\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}-\dfrac{\sin\left(3\alpha+\dfrac{n-1}{2}\cdot 3\beta\right)\sin\dfrac{3n\beta}{2}}{\sin\dfrac{3\beta}{2}}\right\}$.

(vi) $\dfrac{1}{4}\left\{3\dfrac{\sin\dfrac{n+1}{2}\alpha\sin\dfrac{n\alpha}{2}}{\sin\dfrac{1}{2}\alpha}-\dfrac{\sin\dfrac{3(n+1)}{2}\alpha\sin\dfrac{3n\alpha}{2}}{\sin\dfrac{3}{2}\alpha}\right\}$

(vii) $\dfrac{1}{8}\left\{3n-\dfrac{4\cos(n+1)\alpha\sin n\alpha}{\sin\alpha}+\dfrac{\cos 2(n+1)\alpha\sin 2n\alpha}{\sin 2\alpha}\right\}$.

(viii) $\dfrac{1}{2}\left\{\cos 2\theta-\cos 2^{n+1}\theta\right\}$. (ix) $\dfrac{1}{4.3^{n-1}}\left\{3^n\sin\alpha-\sin 3^n\alpha\right\}$

(x) $\dfrac{1}{2}\left\{\dfrac{\sin n\alpha}{\sin\alpha}\sin(n+p+1)\alpha+n\sin p\alpha\right\}$.

(xi) 0.

(xii) $\dfrac{n}{2}\cos\theta+\dfrac{1}{2}\dfrac{\sin 2n\theta}{\sin 2\theta}\cos(2n+1)\theta$.

(xiii) $\dfrac{1}{2}\cdot\dfrac{\sin\frac{1}{2}n(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)}\cdot\sin\{2x+\frac{1}{2}(n-1)(2x+\pi)\}$.

(xiv) $\dfrac{1}{4}\cdot\dfrac{\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}\{2\sin\frac{1}{2}(n+3)\theta+\sin\frac{1}{2}(n-1)\theta\}$

$$-\frac{1}{4}\frac{\sin\frac{1}{2}(3n+5)\theta}{\sin\frac{3}{2}\theta}\sin\frac{3n\theta}{2}.$$

(xv) $\dfrac{\sin\frac{1}{4}n\alpha}{\sin\frac{1}{4}\alpha}\left\{\sin\dfrac{n+1}{4}\alpha+\cos\dfrac{n+1}{4}\alpha\right\}$.

3. (i) $\dfrac{\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}\sin\dfrac{n+1}{2}\alpha$. (ii) $\dfrac{\sin 2n\alpha}{2\sin\alpha}$.

(iii) $\frac{1}{2}n(n+1)$. (iv) $\frac{1}{8}\{(2n)^2-2n\}$.

4. (i) $\dfrac{1}{4\sin\alpha}\{(n+1)\sin 2\alpha-\sin 2(n+1)\alpha\}$. (ii) $\frac{1}{3}n(n+1)(n+2)$.

6. $\dfrac{n}{2}(\cos\alpha\mp i\sin\alpha)$, for cases namely, $x=\cos\dfrac{2\pi}{n}\pm i\sin\dfrac{2\pi}{n}$.

7. (i) $\pm\frac{1}{4}\pi$. (ii) $\dfrac{4\sin\alpha}{5-4\cos\alpha}$. (iii) $e^{\cot\theta}\cos(1+\theta)$.

(iv) $\dfrac{x(\cos\theta-x)}{1-2x\cos\theta+x^2}-\dfrac{1}{2}\log(1-2x\cos\theta+x^2)$.

(v) $\dfrac{\cos \frac{1}{4}\theta}{\sqrt{2} \cos \frac{1}{2}\theta}$. (vi) $e^{x \cos \beta} \sin (a + x \sin \beta)$.

(vii) $\tan^{-1} \dfrac{\sin^2\theta}{1+\cos\theta \sin\theta}$. (viii) $\dfrac{\cos\theta}{1-\cos\theta} - \log (1-\cos\theta)$.

(ix) $\sin\theta \cos\theta + \theta$. (x) $\frac{1}{2}\theta$. (xi) 0.

(xii) $e^{\cos x} \cos (\sin x)$. (xiii) $\frac{1}{2} \log (1 + 2c \cos a + c^2)$.

(xiv) $\frac{1}{2} \log \dfrac{1+m}{\sqrt{1+2m \cos 2a + m^2}}$.

(xv) $\dfrac{\pi \sin x}{1 + 2\pi \cos x + \pi^2}$. (xvi) $\frac{1}{2}\pi - x$.

8. (i) $\cot a - 2^n \cot 2^n a$. (ii) $\dfrac{1}{\sin\theta} \{\cot\theta - \cot (n+1)\theta\}$.

(iii) $\dfrac{1}{2 \sin a} \{\tan (n+1)a - \tan a\}$. (iv) $\tan^{-1} \dfrac{n}{2+n}$.

(v) $\tan^{-1} \dfrac{n}{n+1}$. (vi) $\dfrac{\sqrt{2} \sin \dfrac{n\beta}{2}}{\sin \dfrac{1}{2}\beta} \sin \left\{a + \dfrac{n-1}{2}\beta + \dfrac{\pi}{4}\right\}$

(vii) $\tan^{-1} \dfrac{n}{n+2}$. (viii) $\tan^{-1} \dfrac{n}{n+1}$.

(ix) $\dfrac{(n+1) \cos n\theta - n \cos (n+1)\theta - 1}{2(1-\cos\theta)}$.

(x) $\dfrac{1}{4}\left[\dfrac{\sin \frac{1}{2}nx}{\sin \frac{1}{2}x} \cos \dfrac{n+3}{2}x \cdot \left\{1 + 2 \cos 2x\right\} + \dfrac{\sin \frac{3}{2}nx}{\sin \frac{3}{2}x} \cos \dfrac{3(n+3)}{2}x\right]$.

(xi) $\cos\theta - \sin\theta \cot 2^n \theta$.

10. $\dfrac{1}{2^{n-1}} \cot \dfrac{x}{2^{n-1}} - 2 \cot 2x$; $\dfrac{1}{x} - 2 \cot 2x$.

11. $\cot x \tan (n+1)x - (n+1)$; $\dfrac{n(n+1)(n+2)}{3}$. 12. $\dfrac{\sin \dfrac{2n-1}{2}\theta}{\sin \dfrac{1}{2}\theta}$.

13. $16 \cos^4\theta - 12 \cos^2\theta + 1$.

CHAPTER XII

HYPERBOLIC FUNCTIONS

79. Definition of Hyperbolic Functions.

In analogy with the exponential values $\frac{1}{2i}(e^{ix}-e^{-ix})$ and $\frac{1}{2}(e^{ix}+e^{-ix})$ of $\sin x$ and $\cos x$, the functions $\frac{1}{2}(e^{x}-e^{-x})$ and $\frac{1}{2}(e^{x}+e^{-x})$ (x being *real* or *complex*) are called the **hyperbolic sine** and **hyperbolic cosine** of x and are written as **sinh x** and **cosh x** respectively ; the other functions are defined thus :

$$\sinh x=\tfrac{1}{2}(e^{x}-e^{-x})\ ;\ \cosh x=\tfrac{1}{2}(e^{x}+e^{-x}).$$

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}};\ \coth x=\frac{1}{\tanh x};$$

$$\operatorname{cosech} x=\frac{1}{\sinh x};\ \operatorname{sech} x=\frac{1}{\cosh x}.$$

Note 1. From the definition, it follows that

$$\mathbf{e^{x}=\cosh x+\sinh x} \text{ and } \mathbf{e^{-x}=\cosh x-\sinh x}.$$

Note 2. Just as the ordinate and abscissa of a point P on the circle $x^2+y^2=a^2$ are denoted by $a\sin\theta$, $a\cos\theta$, so the ordinate and abscissa of a point Q on the rectangnlar hyperbola $x^2-y^2=a^2$ may be denoted by $a\sinh\theta$, $a\cosh\theta$. Thus, the hyperbolic sine and cosine have the same relation to the rectangular hyperbola, as the sine and cosine have to the circle. For this reason the former functions are called **Hyperbolic functions**, just as the latter are called **Circular functions.**

80. Hyperbolic Functions expressed in terms of Circular Functions.

Since, for all values of θ, we have

$$\sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right) \text{ and } \cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

hence, putting ix for θ, it easily follows that

$$\frac{\sin ix}{i} = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x$$

and $$\cos ix = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

Thus, $$\mathbf{\sin ix = i \sinh x}$$

$$\mathbf{\cos ix = \cosh x}$$

$$\mathbf{\tan ix = i \tanh x.}$$

Thus, the hyperbolic functions of x are expressed in terms of circular functions of ix by following equations :

$$\sinh x = -i \sin ix \; ; \; \cosh x = \cos ix$$

$$\operatorname{cosech} x = i \operatorname{cosec} ix \; ; \; \operatorname{sech} x = \sec ix$$

$$\tanh x = -i \tan ix \; ; \; \coth x = i \cot ix.$$

81. Formulæ of Hyperbolic Functions.

In general corresponding to most of the trigonometrical formulæ involving circular functions, there are formulæ involving hyperbolic functions. A list of some such formulæ is given below

$$\mathbf{\cosh^2 x - \sinh^2 x = 1} \quad \cdots \quad (1)$$

$$\mathbf{\operatorname{sech}^2 x + \tanh^2 x = 1} \quad \cdots \quad (2)$$

$$\mathbf{\coth^2 x - \operatorname{cosech}^2 x = 1} \quad \cdots \quad (3)$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \quad \cdots \quad (4)$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \quad \cdots \quad (5)$$

$$\tanh (x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \quad \cdots \quad \cdots \quad (6)$$

$$\sinh 2x = 2 \sinh x \cosh x \quad \cdots \quad \cdots \quad (7)$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2 \sinh^2 x \quad (8)$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \quad \cdots \quad \cdots \quad (9)$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y) \quad \cdots \quad (10)$$

$$\sinh x - \sinh y = 2\cosh \tfrac{1}{2}(x+y)\sinh \tfrac{1}{2}(x-y) \quad \cdots (11)$$

$$\cosh x + \cosh y = 2\cosh \tfrac{1}{2}(x+y)\cosh \tfrac{1}{2}(x-y) \quad \cdots (12)$$

$$\cosh x - \cosh y = 2\sinh \tfrac{1}{2}(x+y)\sinh \tfrac{1}{2}(x-y) \quad \cdots (13)$$

$$2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y) \quad \cdots (14)$$

$$2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y) \quad \cdots (15)$$

$$2\cosh x\cosh y = \cosh(x+y) + \cosh(x-y) \quad \cdots (16)$$

$$2\sinh x\sinh y = \cosh(x+y) - \cosh(x-y) \quad \cdots (17)$$

$$\sinh 0 = 0\ ;\ \cosh 0 = 1\ ;\ \tanh 0 = 0 \quad \cdots (18)$$

$$\sinh(-x) = -\sinh x\ ;\ \cosh(-x) = \cosh x \quad \cdots (19)$$

82. The above formulæ can be deduced either from the corresponding formulæ of the circular functions or they may be obtained directly from the definitions.

Let us consider the first formula.

Since, for all values of θ, we have

$$\cos^2\theta + \sin^2\theta = 1$$

hence, putting ix for θ, we have,

$$\cos^2 ix + \sin^2 ix = 1.$$

$\therefore$ by Art. 80, $\cosh^2 x - \sinh^2 x = 1.$

Otherwise :

$$\cosh^2 x - \sinh^2 x = \{\tfrac{1}{2}(e^x + e^{-x})\}^2 - \{\tfrac{1}{2}(e^x - e^{-x})\}^2$$

$$= \tfrac{1}{4}\{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2\}$$

$$= 1.$$

Similarly, the other formulæ can be established.

83. By means of the first three formulæ combined with the definitions of hyperbolic functions, any one hyperbolic function can be expressed in terms of any other. Thus,

if $\sinh x = u$, then $\cosh x = \sqrt{1+u^2}$, $\tanh x = \dfrac{u}{\sqrt{1+u^2}}$ and $\operatorname{cosech} x$, $\operatorname{sech} x$, $\coth x$ are respectively equal to the reciprocals of these values. Similarly, all the hyperbolic functions can be expressed in terms of $\cosh x$, $\tanh x$ etc.

Note. Just as $\sqrt{1-x^2}$ can be expressed in a rational form by putting $x = \sin\theta$, or $\cos\theta$, so $\sqrt{1+x^2}$ and $\sqrt{x^2-1}$ can be expressed in rational forms by putting $x = \sinh\theta$ and $\cosh\theta$ respectively. These substitutions are of great use in integration.

84. Expansions of sinh x and cosh x.

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$= \frac{1}{2}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\cdots\cdots$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$= \frac{1}{2}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\cdots\cdots$$

85. Periodicity of Hyperbolic functions.

$$\sinh(\theta + 2in\pi) = \tfrac{1}{2}[e^{\theta+2in\pi} - e^{-(\theta+2in\pi)}]$$

$$= \tfrac{1}{2}(e^\theta - e^{-\theta}),$$

since $e^{2in\pi} = \cos 2n\pi + i\sin 2n\pi = 1$, n being *any integer* ;

$\therefore\quad \sinh(\theta + 2in\pi) = \sinh\theta.$

Similarly, $\cosh(\theta + 2in\pi) = \cosh\theta$.

It can be shown, as before,

$$\sinh(\theta + in\pi) = -\sinh\theta\,;\ \cosh(\theta + in\pi) = -\cosh\theta.$$

$\therefore\quad \tanh(\theta + in\pi) = \tanh\theta.$

Thus, the hyperbolic functions are *periodic* functions with an *imaginary period* ; the period of $\tanh x$ is half that of $\sinh x$ and $\cosh x$.

86. Gudermannian Function.*

If $\cosh u = \sec\theta$, then $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1}$
$= \tan\theta$.

Conversely, if $\sinh u = \tan\theta$, then $\cosh u = \sqrt{1+\sinh^2 u}$
$= \sqrt{1+\tan^2\theta} = \sec\theta$.

$$\therefore \quad \sec\theta + \tan\theta = \cosh u + \sinh u$$
$$= \tfrac{1}{2}(e^u + e^{-u}) + \tfrac{1}{2}(e^u - e^{-u}) = e^u.$$

$$\therefore \quad u = \log(\sec\theta + \tan\theta)$$
$$= \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta), \text{ or } \log\cot(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta),$$

from elementary trigonometry.

θ is usually called the *Gudermannian function* † of u and is denoted by $gd\ u$;

so that $\theta = gd\ u = \sec^{-1}(\cosh u)$
$= \tan^{-1}(\sinh u)$

and $u = gd^{-1}\theta = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)$.

87. Inverse Hyperbolic Functions.

If $\sinh u = z$, then u is called the *inverse hyperbolic* sine of z and is denoted by $\sinh^{-1}z$. A similar definition applies to $\cosh^{-1}z$, $\tanh^{-1}z$, $\operatorname{cosech}^{-1}z$, $\operatorname{sech}^{-1}z$ and $\coth^{-1}z$.

* Called after the name of the mathematician *C. Gudermann* (1798-1851), whose researches on hyperbolic functions led to the discovery of the function

$\tan^{-1}(\sinh x)$.

† Some mathematicians call θ as the *hyperbolic amplitude of u.*

(i) Let $\sinh^{-1}z = u$,

then $z = \sinh u = \frac{1}{2}(e^u - e^{-u}) = \frac{1}{2}\left[e^u - \frac{1}{e^u}\right]$.

$$\therefore \quad e^{2u} - 2z.e^u - 1 = 0.$$

Solving this as a quadratic for e^u,

$$e^u = \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1}.$$

$$\therefore \quad u = 2n\pi i + \log(z \pm \sqrt{z^2+1}).$$

Noting that

$$z - \sqrt{z^2+1} = \frac{(z-\sqrt{z^2+1})(z+\sqrt{z^2+1})}{z+\sqrt{z^2+1}} = \frac{-1}{z+\sqrt{z^2+1}}$$

and that $\log(-1) = (2k+1)\pi i$,

both values of u can be included in the expression

$$u = in\pi + (-1)^n \log(z + \sqrt{z^2+1}).$$

Thus, the *general value* of $\sinh^{-1}z$

$$= in\pi + (-1)^n \log(z + \sqrt{z^2+1})$$

and its *principal value* is $\log(z + \sqrt{z^2+1})$.

(ii) Let $\cosh^{-1}z = u$,

then $z = \cosh u = \frac{1}{2}(e^u + e^{-u}) = \frac{1}{2}\left[e^u + \frac{1}{e^u}\right]$

$$\therefore \quad e^{2u} - 2z.e^u + 1 = 0.$$

$\therefore$ as before, $e^u = z \pm \sqrt{z^2-1}$,

$$\therefore \quad u = 2n\pi i + \log(z \pm \sqrt{z^2-1}).$$

Noting that $z - \sqrt{z^2-1} = \dfrac{1}{z+\sqrt{z^2-1}}$;

both values of u can be included in the expression

$$u = 2n\pi i \pm \log (z + \sqrt{z^2 - 1}).$$

Thus, the *general value* of $\cosh^{-1} z = 2n\pi i \pm \log (z + \sqrt{z^2 - 1})$ and its *principal value* is $\log (z + \sqrt{z^2 - 1})$.

(iii) Let $\tanh^{-1} z = u$,

then, $z = \tanh u = \dfrac{\sinh u}{\cosh u} = \dfrac{e^u - e^{-u}}{e^u + e^{-u}} = \dfrac{e^{2u} - 1}{e^{2u} + 1}$.

$$\therefore \quad e^{2u} = \frac{1 + z}{1 - z};$$

$$\therefore \quad 2u = 2n\pi i + \log \frac{1 + z}{1 - z}.$$

$\therefore$ u *i.e.*, *general value* of $\tanh^{-1} z = n\pi i + \dfrac{1}{2} \log \dfrac{1 + z}{1 - z}$

and its *principal value* is $\dfrac{1}{2} \log \dfrac{1 + z}{1 - z}$.

By $\sinh^{-1} z$, $\cosh^{-1} z$, $\tanh^{-1} z$ are usually meant their principal values.

Similarly, the *principal values* of $\operatorname{cosech}^{-1} z$, $\operatorname{sech}^{-1} z$ and $\coth^{-1} z$ are respectively the expressions

$$\log \frac{1 + \sqrt{1 + z^2}}{z}, \quad \log \frac{1 + \sqrt{1 - z^2}}{z}, \quad \frac{1}{2} \log \frac{z + 1}{z - 1}.$$

88. Inverse Hyperbolic Functions expressed as Inverse Circular Functions.

If $\quad z = \sinh u, \quad \cdots \quad \cdots \quad (1)$

then, $z = -i \sin iu$, since, $\sinh u = -i \sin iu$.

$\therefore \; iz = \sin iu.$

$$\therefore \quad u = \frac{1}{i} \sin^{-1}(iz) = -i \sin^{-1}(iz).$$

But from (1), $\quad u = \sinh^{-1} z.$

$$\therefore \quad \sinh^{-1} z = -i \sin^{-1}(iz).$$

Similarly, $\cosh^{-1} z = -i \cos^{-1}(z)$

$$\tanh^{-1} z = -i \tan^{-1}(iz).$$

By means of the expressions obtained for the inverse circular functions of a complex argument, the value of the inverse hyperbolic functions can be easily obtained.

89. **Ex. 1.** *Separate the following expressions into their real and imaginary parts (x and y being real)* :

(*i*) *sin* $(x+iy)$. [*C. P. 1942, '50.*] (*ii*) *cos* $(x+iy)$. [*C. P. 1931.*]

(*iii*) *tan* $(x+iy)$. [*P. P. 1931.*] (*iv*) *sinh* $(x+iy)$.

(i) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$$= \sin x \cosh y + i \cos x \sinh y, \text{ by Art. 80.}$$

(ii) Similarly, $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

(iii) $\tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)}$

$$= \frac{\sin(x+iy)\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

(iv) $\sinh(x+iy) = -i \sin\{i(x+iy)\}$, by Art. 80

$$= -i \sin(ix - y)$$

$$= -i(\sin ix \cos y - \cos ix \sin y)$$

$$= -i \sin ix \cos y + i \cos ix \sin y$$

$$= \sinh x \cos y + i \cosh x \sin y.$$

Ex. 2. *Separate into real and imaginary parts the expression*

$$\tan^{-1}(x+iy). \qquad \text{[C. H. 1934, '40, '46.]}$$

Let $\tan^{-1}(x+iy)=\alpha+i\beta$, so that, $\tan(\alpha+i\beta)=x+iy$;

then, $\tan(\alpha-i\beta)=x-iy$, by Art. 47 or as can be shown by using (iii) of Ex. 1, above, and equating real and imaginary parts.

$$\therefore \quad \tan 2\alpha=\tan\{(\alpha+i\beta)+(\alpha-i\beta)\}$$

$$=\frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$=\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}=\frac{2x}{1-x^2-y^2};$$

$$\therefore \quad \alpha=\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}. \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

$$\tan(2i\beta)=\tan\{(\alpha+i\beta)-(\alpha-i\beta)\}$$

$$=\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}=\frac{2iy}{1+x^2+y^2}.$$

Since, $\tan(2i\beta)=i\tanh 2\beta$,

$$\therefore \quad i\tanh 2\beta=\frac{2iy}{1+x^2+y^2};$$

$$\therefore \quad \beta=\frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}. \qquad \cdots \qquad \cdots \qquad \text{(ii)}$$

Hence, from (i) and (ii),

$$\tan^{-1}(x+iy)=\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}+\frac{i}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}.$$

Ex. 3. *Express* $\sinh 2\alpha+\sinh 2\beta+\sinh 2\gamma-\sinh(2\alpha+2\beta+2\gamma)$ *in factors.*

$$\text{Given exp.}=(\sinh 2\alpha+\sinh 2\beta)+\{\sinh 2\gamma-\sinh(2\alpha+2\beta+2\gamma)\}$$

$$=2\sinh(\alpha+\beta)\cosh(\alpha-\beta)-2\cosh(\alpha+\beta+2\gamma)\sinh(\alpha+\beta)$$

$$=2\sinh(\alpha+\beta)\{\cosh(\alpha-\beta)-\cosh(\alpha+\beta+2\gamma)\}$$

$$=2\sinh(\alpha+\beta)\{-2\sinh(\alpha+\gamma)\sinh(\beta+\gamma)\}$$

$$=-4\sinh(\alpha+\beta)\sinh(\beta+\gamma)\sinh(\gamma+\alpha).$$

Ex. 4. *Express $\cosh^7\theta$ in terms of hyperbolic cosines of multiples of θ.*

$$\cosh^7\theta = \frac{1}{2^7}\,(e^\theta + e^{-\theta})^7$$

$$= \frac{1}{2^7}\,[e^{7\theta} + 7e^{5\theta} + 21e^{3\theta} + 35e^\theta + 35e^{-\theta} + 21e^{-3\theta} + 7e^{-5\theta} + e^{-7\theta}]$$

$$= \frac{1}{2^7}[(e^{7\theta} + e^{-7\theta}) + 7(e^{5\theta} + e^{-5\theta}) + 21(e^{3\theta} + e^{-3\theta}) + 35(e^\theta + e^{-\theta})]$$

$$= \frac{1}{2^7}\,[2\cosh 7\theta + 7.2\cosh 5\theta + 21.2\cosh 3\theta + 35.2\cosh\theta]$$

$$= \frac{1}{2^6}\,[\cosh 7\theta + 7\cosh 5\theta + 21\cosh 3\theta + 35\cosh\theta].$$

Ex. 5. *If $\sin(\theta + i\phi) = \tan(x + iy)$, show that*

$$\frac{\tan\theta}{\tanh\phi} = \frac{\sin 2x}{\sinh 2y}.$$

Since, $\sin(\theta + i\phi) = \tan(x + iy)$,

$\therefore\ \sin(\theta - i\phi) = \tan(x - iy)$; [see *Art. 47, Theo. II*]

$$\therefore\ \frac{\sin(\theta + i\phi) + \sin(\theta - i\phi)}{\sin(\theta + i\phi) - \sin(\theta - i\phi)} = \frac{\tan(x + iy) + \tan(x - iy)}{\tan(x + iy) - \tan(x - iy)};$$

$$\therefore\ \frac{2\sin\theta\cos(i\phi)}{2\cos\theta\sin(i\phi)} = \frac{\sin\{(x + iy) + (x - iy)\}}{\sin\{(x + iy) - (x - iy)\}};$$

$$\therefore\ \frac{\tan\theta}{\tan i\phi} = \frac{\sin 2x}{\sin(2iy)}. \qquad \therefore\ \frac{\tan\theta}{i\tanh\phi} = \frac{\sin 2x}{i\sinh 2y}.$$

Hence, the result.

Ex. 6. *Show that $\tan^{-1}(\cot\theta\tanh\phi) = \frac{1}{2i}\log\frac{\sin(\theta + i\phi)}{\sin(\theta - i\phi)}$.*

Let $\quad \tan^{-1}(\cot\theta\tanh\phi) = z.$

Then, $\quad \tan z = \cot\theta\tanh\phi$

or, $$\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = i.\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}\cdot\frac{e^\phi - e^{-\phi}}{e^\phi + e^{-\phi}}$$

or, $$\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}=-\frac{(e^{i\theta}+e^{-i\theta})(e^{\phi}-e^{-\phi})}{(e^{i\theta}-e^{-i\theta})(e^{\phi}+e^{-\phi})}.$$

$\therefore$ by componendo-dividendo,

$$\frac{e^{iz}}{e^{-iz}}=\frac{e^{i\theta-\phi}-e^{-i\theta+\phi}}{e^{i\theta+\phi}-e^{-i\theta-\phi}}$$

or, $$e^{2iz}=\frac{e^{i(\theta+i\phi)}-e^{-i(\theta+i\phi)}}{e^{i(\theta-i\phi)}-e^{-i(\theta-i\phi)}}$$

$$=\frac{\sin(\theta+i\phi)}{\sin(\theta-i\phi)}.$$

$\therefore$ taking principal values of both sides

$$2iz=\log\frac{\sin(\theta+i\phi)}{\sin(\theta-i\phi)}$$

whence the result follows.

Ex. 7. *Find the sum to infinity of*

$$\cosh x+\frac{\sin x}{1!}\cosh 2x+\frac{\sin^2 x}{2!}\cosh 3x+\cdots$$ [*C. H. 1939.*]

Let S denote the sum; then writing $\cosh x=\frac{1}{2}(e^x+e^{-x})$,

$\cosh 2x=\frac{1}{2}(e^{2x}+e^{-2x})$, etc., and putting $a=e^x$, $b=e^{-x}$, we get

$$2S=\left[a+\frac{\sin x}{1!}a^2+\frac{\sin^2 x}{2!}a^3+\cdots\right]+\left[b+\frac{\sin x}{1!}b^2+\frac{\sin^2 x}{2!}b^3+\cdots\right]$$

$$=ae^{a\sin x}+be^{b\sin x}$$

$$=e^x e^{e^x\sin x}+e^{-x}e^{e^{-x}\sin x}$$

$$=e^x e^{\sin x(\sinh x+\cosh x)}+e^{-x}e^{\sin x(\cosh x-\sinh x)}$$

$$=e^{\sin x\cosh x}\left[e^{x+\sin x\sinh x}+e^{-x-\sin x\sinh x}\right],$$

$$=e^{\sin x\cosh x}[\,2\cosh(x+\sin x\sinh x)\,].$$

EXAMPLES XII

1. Prove that :—

(i) $\sinh 3x = 4\sinh^3 x + 3\sinh x$.

(ii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

(iii) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$.

(iv) $\tanh(x+y+z)$

$$= \frac{\tanh x + \tanh y + \tanh z + \tanh x \tanh y \tanh z}{1 + \tanh y \tanh z + \tanh z \tanh x + \tanh x \tanh y}.$$

(v) $\sinh^{-1} x + \sinh^{-1} y = \sinh^{-1}(x\sqrt{y^2+1} + y\sqrt{x^2+1})$.

(vi) $\tanh^{-1} x \pm \tanh^{-1} y = \tanh^{-1} \dfrac{x \pm y}{1 \pm xy}$.

(vii) $\sinh^{-1} x = \cosh^{-1}\sqrt{(1+x^2)} = \tanh^{-1} \dfrac{x}{\sqrt{(1+x^2)}}$.

(viii) $(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx$.

[*C. H. 1922.*]

(ix) $(\cosh\theta_1 + \sinh\theta_1)(\cosh\theta_2 + \sinh\theta_2)\ldots\ldots$

$\ldots(\cosh\theta_n + \sinh\theta_n)$

$$= \cosh(\theta_1 + \theta_2 + \cdots + \theta_n) + \sinh(\theta_1 + \theta_2 + \cdots + \theta_n).$$

(x) $\dfrac{\cosh\theta + \sinh\theta}{\cosh\phi + \sinh\phi} = \cosh(\theta - \phi) + \sinh(\theta - \phi)$.

(xi) $\dfrac{\sinh(u-v) + \sinh u + \sinh(u+v)}{\cosh(u-v) + \cosh u + \cosh(u+v)} = \tanh u$.

(xii) $(1 + \cosh\theta + \sinh\theta)^n$

$$2^n \cosh^n \frac{\theta}{2}\left(\cosh\frac{n\theta}{2} + \sinh\frac{n\theta}{2}\right)$$

2. Separate into real and imaginary parts :

(i) $\cosh(x+iy)$. (ii) $\tanh(x+iy)$.

(iii) $\sec(x+iy)$. [*C. H. 1942.*]

(iv) (*a*) $\sin^{-1}(\cos\theta+i\sin\theta)$. [*P. H. 1935.*]

(*b*) $\tan^{-1}(\cos\theta+i\sin\theta)$, $0<\theta<\frac{1}{2}\pi$.

(v) $\sin^{-1}(x+iy)$. (vi) $\cos^{-1}(x+iy)$.

(vii) $\{\cos(\theta+i\phi)+i\sin(\theta-i\phi)\}^{\alpha+i\beta}$.

(viii) $\log\{\operatorname{cosec}(x+iy)\}$. [*C. H. 1942.*]

3. If $\cos^{-1}(\alpha+i\beta)=\theta+i\phi$, show that

$$\alpha^2\operatorname{sech}^2\phi+\beta^2\operatorname{cosech}^2\phi=1.$$

4. If $\sin(\alpha+i\beta)=x+iy$, prove that

(i) $x^2\operatorname{cosec}^2\alpha-y^2\sec^2\alpha=1$.

(ii) $x^2\operatorname{sech}^2\beta+y^2\operatorname{cosech}^2\beta=1$.

5. If $\tan(\alpha+i\beta)=x+iy$, prove that

(i) $x^2+y^2+2x\cot2\alpha=1$.

(ii) $x^2+y^2-2y\coth2\beta=-1$. [*P. H. 1934.*]

6. If $\tan(x+iy)=\cos\alpha+i\sin\alpha$, prove that

$$e^{2y}=\pm\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha) \text{ and } x=\tfrac{1}{2}n\pi\pm\tfrac{1}{4}\pi.$$

[*C. H. 1928.*]

7. Show that the general solution of

(i) $\sinh x=1$

is $x=in\pi+(-1)^n\log(1+\sqrt{2})$. [*C. H. 1920.*]

(ii) $\tan^{-1}(e^{ix})-\tan^{-1}(e^{-ix})=\tan^{-1}i$

is $x=n\pi+(-1)^n\frac{1}{2}\pi$. [*C. H. 1926.*]

8. If $x=\cos\theta-\frac{1}{3}\cos3\theta+\frac{1}{5}\cos5\theta-\cdots$,

and $y=\sin\theta-\frac{1}{3}\sin3\theta+\frac{1}{5}\sin5\theta-\cdots$

show that $\tan\theta=\sinh2y\operatorname{cosec}2x$.

9. If $\cosh^{-1}(x+iy)+\cosh^{-1}(x-iy)=\cosh^{-1}z$,

$$\text{then } 2(z-1)x^2+2(z+1)y^2=z^2-1.$$

10. If $\cosh x=\sec\theta$ and $x=\theta+a_3\theta^3+a_5\theta^5+\cdots$

$$\text{prove that } \theta=x-a_3x^3+a_5x^5-\cdots$$

11. Show that

(i) $\tan^{-1}\left[\dfrac{\tan 2x+\tanh 2y}{\tan 2x-\tanh 2y}\right]+\tan^{-1}\left[\dfrac{\tan x-\tanh y}{\tan x+\tanh y}\right]$

$$=\tan^{-1}(\cot x\coth y).$$

(ii) $\tanh^{-1}x=x+\frac{1}{3}x^3+\frac{1}{5}x^5+\cdots$ to infinity.

(iii) $\tanh\theta+\frac{1}{3}\tanh^3\theta+\frac{1}{5}\tanh^5\theta+\cdots$

$$=\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\cdots$$

where θ lies between $\pm\frac{1}{4}\pi$.

12. Sum the following series :

(i) $\sinh x+\sinh(x+y)+\sinh(x+2y)+\cdots$ to n terms.

(ii) $\cosh x+\cosh(x+y)+\cosh(x+2y)+\cdots$ to n terms.

(iii) $\sinh x-\frac{1}{2}\sinh 2x+\frac{1}{3}\sinh 3x-\cdots$ to inf.

(iv) $1+\cosh x+\dfrac{1}{2\,!}\cosh 2x+\dfrac{1}{3\,!}\cosh 3x+\cdots$ to inf.

(v) $\tanh^{-1}x+\tanh^{-1}\dfrac{x}{1-1.2x^2}$

$$+\tanh^{-1}\frac{x}{1-2.3x^2}+\cdots \text{ to } n \text{ terms.}$$

(vi) $\sinh x\cosh 2x+\sinh 3x\cosh 4x$

$+\sinh 5x\cosh 6x+\cdots$ to n terms. [*C. H. 1938.*]

(vii) $\cosh^2 x+\cosh^2 3x+\cosh^2 5x+\cdots$ to n terms.

(viii) $\cosh\theta+2\cosh 2\theta+3\cosh 3\theta+\cdots$ to n terms.

(ix) $1 + x \cosh \theta + x^2 \cosh 2\theta + x^3 \cosh 3\theta + \cdots$ to inf.

(x) $x \cosh \alpha + \dfrac{x^2}{2!} \cosh 2\alpha + \dfrac{x^3}{3!} \cosh 3\alpha + \cdots$ to inf.

[*C. P. 1937.*]

13. Solve :

(i) $\cosh (\log x) = \sinh (\log \frac{1}{2}x) + \frac{7}{4}$.

(ii) $\cos (x + iy) = \frac{5}{4}$.

14. If $x > 0$, show that

$$\cosh x > \sinh x > x > \tanh x.$$

15. Show that

$$\underset{x \to 0}{Lt} \frac{\sinh x}{x} = 1 \; ; \; \underset{x \to 0}{Lt} \frac{x}{\tanh x} = 1.$$

16. Evaluate $\underset{\theta \to 0}{Lt} \dfrac{\sinh \theta - \sin \theta}{\theta^3}$.

17. If $\cos \theta \cosh \phi = \cos \alpha$, $\sin \theta \sinh \phi = \sin \alpha$,

prove that $\sin \alpha = \pm \sin^2 \theta = \pm \sinh^2 \phi$.

18. If $\cos^{-1} (u + iv) = \alpha + i\beta$, prove that

$\cos^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$ [*C. H. 1938.*]

19. If $\sin^{-1} (u + iv) = \alpha + i\beta$, prove that

$\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$

20. If $\tan^m (\frac{1}{4}\pi + \frac{1}{2}\phi) = \tan^n (\frac{1}{4}\pi + \frac{1}{2}\psi)$,

show that $m \tan^{-1} \left(\dfrac{\sin \phi}{i}\right) = n \tan^{-1} \left(\dfrac{\sin \psi}{i}\right)$.

ANSWERS

2. (i) $\cosh x \cos y + i \sinh x \sin y$.

(ii) $\dfrac{\sinh 2x + i \sin 2y}{\cosh 2x + \cos 2y}$. (iii) $2\dfrac{\cos x \cosh y + i \sin x \sinh y}{\cos 2x + \cosh 2y}$.

(iv) (*a*) $x + iy$ where $x = \cos^{-1}(\sqrt{\sin\theta})$,

and $y = \log[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}]$.

(*b*) $\dfrac{\pi}{4} + \dfrac{i}{4}\log\dfrac{1+\sin\theta}{1-\sin\theta}$, or, $\dfrac{\pi}{4} + i.\tfrac{1}{2}\tanh^{-1}(\sin\theta)$.

(v) $n\pi + (-1)^{-n}\sin^{-1} v + (-1)^n i \log(u + \sqrt{u^2+1})$,

where $u = \frac{1}{2}\sqrt{(x+1)^2+y^2} + \frac{1}{2}\sqrt{(x-1)^2+y^2}$,

and $v = \frac{1}{2}\sqrt{(x+1)^2+y^2} - \frac{1}{2}\sqrt{(x-1)^2+y^2}$

(vi) $2n\pi \pm \{\cos^{-1} v - i\log(u + \sqrt{u^2-1})\}$,

where $u = \frac{1}{2}\sqrt{(x+1)^2+y^2} + \frac{1}{2}\sqrt{(x-1)^2+y^2}$,

and $v = \frac{1}{2}\sqrt{(x+1)^2+y^2} - \frac{1}{2}\sqrt{(x-1)^2+y^2}$.

(vii) $e^{\alpha\log r - \beta(\odot + 2n\pi)}[\cos\{\beta\log r + \alpha(\odot + 2n\pi)\}$
$+ i\sin\{\beta\log r + \alpha(\odot + 2n\pi)\}]$,

where $r^2 = e^{2\phi}\cos^2\theta + e^{-2\phi}\sin^2\theta$,

and $\odot = \tan^{-1}(e^{-2\phi}\tan\theta)$.

(viii) $-\frac{1}{2}\log(\sin^2 x + \sinh^2 y) - i\tan^{-1}(\cot x\tanh y)$.

12. (i) $\dfrac{\sinh\left(x + \frac{n-1}{2}y\right)\sinh\frac{ny}{2}}{\sinh\frac{y}{2}}$. (ii) $\dfrac{\cosh\left(x + \frac{n-1}{2}y\right)\sinh\frac{ny}{2}}{\sinh\frac{y}{2}}$

(iii) $\frac{1}{2}x$. (iv) $e^{\cosh x}\cosh(\sinh x)$. (v) $\tanh^{-1} nx$.

(vi) $\dfrac{1}{2}\dfrac{\sinh 2nx}{\sinh 2x}\sinh(2n+1)x - \dfrac{n}{2}\sinh x$. (vii) $\dfrac{n}{2} + \dfrac{\sinh 4nx}{4\sinh 2x}$.

(viii) $\frac{1}{2}\operatorname{cosech}^2\frac{1}{2}\theta\{n\sinh\frac{1}{2}\theta\sinh(n+\frac{1}{2})\theta - \sinh^2\frac{1}{2}n\theta\}$.

(ix) $\dfrac{1 - x\cosh\theta}{1 - 2x\cosh\theta + x^2}$. (x) $\frac{1}{2}\{e^{xe^{\alpha}} + e^{xe^{-\alpha}} - 2\}$.

13. (i) $x = 1$ or 6. (ii) $x = 2n\pi$, $y = \pm\log 2$. **16.** $\frac{1}{3}$.

MISCELLANEOUS EXAMPLES II

[*On Chapters VI-XII*]

1. If α, β are the roots of $t^2 - 2t + 2 = 0$, prove that

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\sin n\theta}{\sin^n \theta}, \text{ where } \theta = \cot^{-1}(x+1).$$

2. If $a = \cos 2\alpha + i \sin 2\alpha$, $b = \cos 2\beta + i \sin 2\beta$,

$$c = \cos 2\gamma + i \sin 2\gamma,\ d = \cos 2\delta + i \sin 2\delta,$$

show that

(i) $\sqrt{abcd} + \dfrac{1}{\sqrt{abcd}} = 2 \cos (\alpha + \beta + \gamma + \delta)$.

(ii) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2 \cos (\alpha + \beta - \gamma - \delta)$.

3. Given $P = \cos 5\theta + i \sin 5\theta$

$$Q = \cos 7\theta + i \sin 7\theta$$

$$R = \cos 11\theta + i \sin 11\theta$$

$$S = \cos 13\theta + i \sin 13\theta$$

show that

(1) S is a *fourth proportional to* P, Q, R.

(2) *One* of the values of the expression

$$P^{\frac{2}{5}} - Q^{\frac{3}{7}} + R^{\frac{3}{11}} - S^{\frac{2}{13}}$$

vanishes for all values of θ. [*C. H. 1926.*]

4. If the product (P) of the n binomials

$$\cos \alpha_1 + i \sin \alpha_1, \cos \alpha_2 + i \sin \alpha_2, \ldots \cos \alpha_n + i \sin \alpha_n$$

be an imaginary cube root of unity, prove that

$$\frac{3(\alpha_1 + \alpha_2 + \cdots\cdots + \alpha_n)}{2\pi}$$

must be an integer of the form $3k+1$, or, $3k+2$, k being zero or any integer positive or negative. [*C. H. 1927.*]

5. Show that $\{\cos x + \cos y + i\,(\sin x + \sin y)\}^n$

$$+\{\cos x + \cos y - i\,(\sin x + \sin y)\}^n$$
$$= 2^{n+1}\,\{\cos \tfrac{1}{2}(x-y)\}^n \cos n.\tfrac{1}{2}(x+y). \quad [\textit{C. P. 1949.}]$$

6. Prove that if n be an *odd* integer, $\sin n\theta + \cos n\theta$ is divisible by $\sin\theta + \cos\theta$ or else by $\sin\theta - \cos\theta$.

[*C. H. 1940.*]

7. Adjust the constants a and b in such a way that

$$\underset{x\to 0}{Lt}\ \frac{a\cos x + bx\sin x - 5}{x^4}$$

may exist finitely, and find this limit. [*C. H. 1942.*]

8. Solve for θ

$$(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ldots\ldots$$
$$\cdots(\cos \overline{2r-1}\,\theta + i\sin \overline{2r-1}\,\theta) = 0. \quad [\textit{C. P. 1942.}]$$

9. Show that $\sin^3\theta$

$$= \frac{3}{4}\left[\frac{3^2-1}{3\,!}\theta^3 - \frac{3^4-1}{5\,!}\theta^5 + \cdots + (-1)^{n-1}\frac{3^{2n}-1}{(2n+1)\,!}\theta^{2n+1} + \cdots\right].$$

[*C. P. 1949.*]

10. From the expansion of $\tan n\theta$ in terms of $\tan\theta$, show that if n be an *odd* integer, then

$$1 - \frac{n(n-1)}{2\,!} + \frac{n(n-1)(n-2)(n-3)}{4\,!} -$$
$$= n - \frac{n(n-1)(n-2)}{3\,!} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5\,!} - \cdots$$

and if n be an *even* integer, one of the two series is zero.

11. If S_n denote the sum of the products of $\tan^2\alpha_1$, $\tan^2\alpha_2$,...... taken n together, and s_n denote the sum of the products of $\tan\alpha_1$, $\tan\alpha_2$, taken n together, show that

$$1 + S_1 + S_2 + S_3 + \cdots = (1 - s_2 + s_4 - \cdots)^2 + (s_1 - s_3 + \cdots)^2.$$

[*Square and add relations (3) & (4) of Art. 56.*]

12. Criticise the fallacy :

For all integral values of n, we have

$$e^{2n\pi i}=\cos 2n\pi+i\sin 2n\pi=1.$$

$$\therefore\quad e^{2\pi i}=e^{4\pi i}=e^{6\pi i}=\cdots$$

Raise all the quantities to the power i ; thus

$$e^{-2\pi}=e^{-4\pi}=e^{-6\pi}=\cdots$$

$$\therefore\quad 2\pi=4\pi=6\pi=\cdots$$

$$\therefore\quad 2=4=6=\cdots$$

13. If α, β be the imaginary cube roots of unity, prove that

$$\alpha e^{\alpha x}+\beta e^{\beta x}=-e^{-\frac{x}{2}}\left[\sqrt{3}\sin\frac{x\sqrt{3}}{2}+\cos\frac{x\sqrt{3}}{2}\right].$$

14. Show that the principal value of

$$\frac{(x+iy)^{a+ib}}{(x-iy)^{a-ib}}$$

is $\cos 2(a\alpha+b\log r)+i\sin 2(a\alpha+b\log r)$,

where $r=\sqrt{x^2+y^2}$ and $\alpha=\tan^{-1}\dfrac{y}{x}$. [*C. H. 1943.*]

15. If $x^{x^{x\cdots\text{ad. inf}}}=a(\cos\alpha+i\sin\alpha)$,

show that the general value of x is $r(\cos\theta+i\sin\theta)$,

$$\text{where}\qquad \log r=\frac{(2n\pi+\alpha)\sin\alpha+\log a\cos\alpha}{a}$$

$$\text{and}\qquad \theta=\frac{(2n\pi+\alpha)\cos\alpha-\log a\sin\alpha}{a}.$$

16. If x be real, prove that

$$\operatorname{Tan}^{-1}x=\frac{1}{2i}\operatorname{Log}\frac{1+ix}{1-ix}.$$

17. Show that

$$\sum_{r=1}^{r=\infty} \sin^{-1} \frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}} = \frac{\pi}{2}.$$

Find the sum up to n terms (*Ex. 18 to Ex. 20*) :—

18. $\sec x \sec 2x + \sec 2x \sec 3x + \sec 3x \sec 4x + \cdots$

$$\left[\sec x \sec 2x = \frac{1}{\sin x} (\tan 2x - \tan x). \right]$$

19. $\cot x \cot 2x + \cot 2x \cot 3x + \cot 3x \cot 4x + \cdots$

[$1 + \cot x \cot 2x = \cot x (\cot x - \cot 2x)$.]

20. $\dfrac{\sin 3\theta}{\sin \theta \sin^2 2\theta} + \dfrac{\sin 6\theta}{\sin 2\theta \sin^2 4\theta} + \dfrac{\sin 12\theta}{\sin 4\theta \sin^2 8\theta},$

$$1st\ term = \frac{4\cos^2\theta - 1}{\sin^2 2\theta} = \text{cosec}^2\theta - \text{cosec}^2 2\theta. \Big]$$

21. Find the sum to infinity of the following series (x lying between 0 and 2π)

$$\frac{\cos x}{1.2} + \frac{\cos 2x}{2.3} + \frac{\cos 3x}{3.4} + \cdots\cdots$$

22. Sum to infinity the series whose rth term is

$$\frac{1}{r!} \cos rx \tan^r x$$

[*Apply $C + iS$ method.*]

23. Show that

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{4}{33} + \tan^{-1}\frac{8}{129} + \cdots \text{ to } \infty = \frac{\pi}{4}.$$

$$\left[u_n = \tan^{-1} \frac{2^{n-1}}{1+2^{2n-1}} = \tan^{-1} 2^n - \tan^{-1} 2^{n-1}. \right]$$

24. Solve completely the equation for x

$$x^n + n \cos a . x^{n-1} + \frac{n(n-1)}{1.2} \cos 2a . x^{n-2} + \cdots + \cos na = 0.$$

[*The equation is* $(x + e^{ia})^n + (x + e^{-ia})^n = 0$.]

25. If $\tan(x+iy)=\tan\phi+i\sec\phi$, show that

$$2x=n\pi+\frac{\pi}{2}+\phi \text{ and } 4y=\log\frac{1+\cos\phi}{1-\cos\phi}.$$

26. If $\log\sin(x+iy)=a+ib$ $(0<x<\pi)$, show that

$$a=\tfrac{1}{2}\log(\cosh^2 y-\cos^2 x) \text{ and } b=\tan^{-1}(\cot x\tanh y).$$

27. Show that the equation $\tan z=az$, where a is real, cannot have a solution of the form $x+iy$, where x and y are both finite quantities (other than zero).

28. Show that $(a+ia\tan\theta)^{\log_e a(\sec\theta)-i\theta}$ is a real number.

29. Prove that if $(1+i\tan\alpha)^{1+i\tan\beta}$ can have real values, one of them is $(\sec\alpha)^{\sec^2\beta}$. [*C. H. 1941.*]

30. Prove that if x be not an odd multiple of $\frac{1}{2}\pi$, $\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-\cdots$

$$=\tfrac{1}{2}(\sin x+\tfrac{1}{3}\sin^3 x+\tfrac{1}{5}\sin^5 x+\cdots).$$

31. If $(1-c)\tan\theta=(1+c)\tan\phi$, then each of the series

$$c\sin 2\theta-\tfrac{1}{2}c^2\sin 4\theta+\tfrac{1}{3}c^3\sin 6\theta-\cdots$$

$$\text{and }\ c\sin 2\phi+\tfrac{1}{2}c^2\sin 4\phi+\tfrac{1}{3}c^3\sin 6\phi+\cdots$$

is equal to $\theta-\phi$, where θ and ϕ vanish together, and $c<1$.

32. If $\tan\frac{1}{2}\theta\equiv\left(\dfrac{1+a}{1-a}\right)^{\frac{1}{2}}\tan\frac{1}{2}\phi$, show that

$$\theta=\phi+2\lambda\sin\phi+\frac{2\lambda^2}{2}\sin 2\phi+\frac{2\lambda^3}{3}\sin 3\phi+\cdots$$

where $\lambda=\frac{1}{2}a+(\frac{1}{2}a)^3+2(\frac{1}{2}a)^5+5(\frac{1}{2}a)^7+\cdots$

ANSWERS

7. $a=5,\ b=2\frac{1}{2}\ ;\ -\frac{5}{27}$. **8.** $2n\pi/r^2$.

18. $2\ \mathrm{cosec}\ 2x \sin nx \sec (n+1)x$.

19. $\cot x\,[\cot x-\cot (n+1)x]-n$. **20.** $\mathrm{cosec}^2\theta-\mathrm{cosec}^2(2^n\theta)$.

21. $1-2\sin^2\frac{x}{2}\log\left(2\sin\frac{x}{2}\right)-\frac{\pi-x}{2}\sin x$.

22. $e^{\sin x}\cos(\sin x\tan x)-1$.

24. The roots are given by $\dfrac{\sin\left\{a_1-\frac{2r+1}{2n}\pi\right\}}{\sin\frac{2r+1}{2n}\pi}$, where r has all the integral values from 0 to $n-1$.

CHAPTER XIII

EXPANSIONS OF COS$^n\theta$, SIN$^n\theta$, COS$n\theta$, SIN$n\theta$

90. Let $x = \cos\theta + i\sin\theta$;

then $\quad x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$,

and $\quad x^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta$
$$= \cos n\theta - i\sin n\theta.$$

Therefore, $\quad x^n + \dfrac{1}{x^n} = 2\cos n\theta$; $x^n - \dfrac{1}{x^n} = 2i\sin n\theta$.

Putting $n=1$, $x + \dfrac{1}{x} = 2\cos\theta$ and $x - \dfrac{1}{x} = 2i\sin\theta$.

91. Expansion of $\cos^n\theta$ in a series of cosines of multiples of θ *(n being a positive integer).*

We have,

$$(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$$
$$= x^n + nx^{n-1}\cdot\frac{1}{x} + \frac{n(n-1)}{\lfloor 2}x^{n-2}\cdot\frac{1}{x^2} + \cdots$$
$$+ \frac{n(n-1)}{\lfloor 2}x^2\cdot\frac{1}{x^{n-2}} + nx\cdot\frac{1}{x^{n-1}} + \frac{1}{x^n}. \quad \cdots \quad (1)$$

Now, combining together the first and the last terms, the second and the last but one, and so on of the series (1), we have,

$2^n\cos^n\theta$

$$= \left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{\lfloor 2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots$$
$$= 2\cos n\theta + n.2\cos(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\cdot 2\cos(n-4)\theta + \cdots$$

[*by Art. 90*]

Hence,

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \cdots \qquad (2)$$

Since n is a positive integer, the number of terms in the expansion of $\left(x+\frac{1}{x}\right)^n$ is finite and equal to $(n+1)$ and hence the series (1) and consequently the series (2) are terminating series.

Note. *Last term* in the expansion of $2^{n-1}\cos^n\theta$.

When n is even, there will be a middle term, namely, $\left(\frac{n}{2}+1\right)$ th term in the expansion of $\left(x+\frac{1}{x}\right)^n$ by the Binomial theorem, and this is equal to $\frac{n(n-1)\cdots(\frac{1}{2}n+1)}{\lfloor\frac{1}{2}n}$. Hence, ***when n is even***, the last term of $2^{n-1}\cos^n\theta$ is $\frac{n(n-1)\cdots(\frac{1}{2}n+1)}{2\lfloor\frac{1}{2}n}$. When n is odd, suppose it is equal to $2m+1$; there are two middle terms in the expansion of $\left(x+\frac{1}{x}\right)^n$ namely, $(m+1)$th and $(m+2)$th, and their sum is

$$\frac{n(n-1)\cdots(n-m+1)}{\lfloor m}\left(x+\frac{1}{x}\right).$$

Hence, ***when n is odd***, the last term of $2^{n-1}\cos^n\theta$

$$=\frac{n(n-1)\cdots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)}\cos\theta.$$

92. Expansion of $\sin^n\theta$ in a series of cosines or sines of multiples of θ (*n being a positive integer*).

We have, $\quad (2i\sin\theta)^n = \left(x-\frac{1}{x}\right)^n. \quad \cdots \quad \cdots \quad (1)$

Case I. Let n be **even.**

Then, $\quad i^n = (i^2)^{\frac{n}{2}} = (-1)^{\frac{n}{2}}$

and the last term in the expansion of (1) is $\frac{1}{x^n}$.

The relation (1) therefore becomes

$$2^n.(-1)^{\frac{n}{2}}\sin^n\theta$$

$$=x^n-nx^{n-1}.\frac{1}{x}+\frac{n(n-1)}{\lfloor 2}.x^{n-2}.\frac{1}{x^2}-\cdots$$

$$\cdots+\frac{n(n-1)}{\lfloor 2}.x^2.\frac{1}{x^{n-2}}-nx.\frac{1}{x^{n-1}}+\frac{1}{x^n}$$

$$=\left(x^n+\frac{1}{x^n}\right)-n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+\frac{n(n-1)}{\lfloor 2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)-\cdots$$

$$=2\cos n\theta-n.2\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}.2\cos(n-4)\theta-\cdots$$

Therefore, $\mathbf{(-1)^{\frac{n}{2}}\,2^{n-1}\sin^n\theta}$

$$\mathbf{=\cos n\theta-n\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}\cos(n-4)\theta-\cdots}$$

$$\mathbf{\cdots+(-1)^{\frac{n}{2}}\frac{n(n-1)\cdots\cdots(\frac{1}{2}n+1)}{2\lfloor\frac{1}{2}n}},$$

the last term being determined as in the previous article.

Case II. Let n be **odd.**

Then, $i^n=i.i^{n-1}=i.(i^2)^{\frac{n-1}{2}}=i(-1)^{\frac{n-1}{2}}$

and the last term in the expansion of (1) is $-\frac{1}{x^n}$.

The relation (1) then becomes

$$2^n.i(-1)^{\frac{n-1}{2}}\sin^n\theta$$

$$=x^n-nx^{n-1}.\frac{1}{x}+\frac{n(n-1)}{\lfloor 2}.x^{n-2}.\frac{1}{x^2}-\cdots$$

$$\cdots\cdots-\frac{n(n-1)}{\lfloor 2}.x^2\frac{1}{x^{n-2}}+nx.\frac{1}{x^{n-1}}-\frac{1}{x^n}$$

$$=\left(x^n-\frac{1}{x^n}\right)-n\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+\frac{n(n-1)}{\lfloor 2}\left(x^{n-4}-\frac{1}{x^{n-4}}\right)-\cdots$$

$$=2i\sin n\theta - n.2i\sin(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\cdot 2i\sin(n-4)\theta - \cdots$$

[*by Art. 90*]

Therefore, $(-1)^{\frac{n-1}{2}} 2^{n-1}\sin^n\theta$

$$=\sin n\theta - n.\sin(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\sin(n-4)\theta - \cdots$$

$$\cdots + (-1)^{\frac{n-1}{2}}\frac{n(n-1)\ldots\ldots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)}\sin\theta,$$

the last term being determined as in the previous article.

Ex. 1. *Expand $\sin^7\theta$ in a series of multiples of θ.*

[*C. P. 1940.*]

By Art. 90, we have, $2^7 i^7\sin^7\theta = \left(x-\frac{1}{x}\right)^7$

$$=x^7-7x^5+21x^3-35x+35\cdot\frac{1}{x}-21\cdot\frac{1}{x^3}+7\frac{1}{x^5}-\frac{1}{x^7}$$

$$=\left(x^7-\frac{1}{x^7}\right)-7\left(x^5-\frac{1}{x^5}\right)+21\left(x^3-\frac{1}{x^3}\right)-35\left(x-\frac{1}{x}\right).$$

Since, $i^7=i^6.i=(i^2)^3.i=(-1)^3.i=-i$, hence by Art. 90, we have

$$-2^7.i\sin^7\theta = 2i\sin 7\theta - 7.2i\sin 5\theta + 21.2i\sin 3\theta - 35.2i\sin\theta.$$

$$\therefore\ \sin^7\theta = -\frac{1}{2^6}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right].$$

93. The methods of Arts. 91, 92 may also be used to express $\sin^p\theta.\cos^q\theta$ in a series of multiples of sines and cosines of θ when p and q are positive integers. This is illustrated in the following example.

Ex. 2. *Expand $\sin^5\theta\cos^2\theta$ in a series of multiples of θ.*

We have, $2^7.i^5\sin^5\theta\cos^2\theta = \left(x-\frac{1}{x}\right)^5\left(x+\frac{1}{x}\right)^2$

$$=\left(x-\frac{1}{x}\right)^3\left(x^2-\frac{1}{x^2}\right)^2$$

$$=\left(x^3-3x+\frac{3}{x}-\frac{1}{x^3}\right)\left(x^4-2+\frac{1}{x^4}\right)$$

$$=\left(x^7-\frac{1}{x^7}\right)-3\left(x^5-\frac{1}{x^5}\right)+\left(x^3-\frac{1}{x^3}\right)+5\left(x-\frac{1}{x}\right)$$

$$=2i\,[\sin 7\theta-3\sin 5\theta+\sin 3\theta+5\sin\theta].$$

$$\therefore\quad \sin^5\theta\cos^2\theta=\frac{1}{2^6}\Big[\sin 7\theta-3\sin 5\theta+\sin 3\theta+5\sin\theta\Big].$$

94. Expansion of cos nθ in *descending* powers of cos θ *(n being a positive integer).*

Let $x=\cos\theta+i\sin\theta$; then $x+\dfrac{1}{x}=2\cos\theta$

and $\quad x^n+\dfrac{1}{x^n}=2\cos n\theta.$

Now, $(1-kx)\left(1-\dfrac{k}{x}\right)=1-k\left(x+\dfrac{1}{x}\right)+k^2$

$=1-ka+k^2,\ \left[\text{putting } a \text{ for } x+\dfrac{1}{x}=2\cos\theta\right]$

$=1-k(a-k).$

Taking logarithms of both sides, we have

$$\log(1-kx)+\log\left(1-\frac{k}{x}\right)=\log\left\{1-k(a-k)\right\}\cdots(1)$$

Hence, expanding both sides of (1), we have

$$kx+\frac{1}{2}k^2x^2+\frac{1}{3}k^3x^3+\cdots+\frac{k}{x}+\frac{1}{2}\frac{k^2}{x^2}+\frac{1}{3}\frac{k^3}{x^3}+\cdots$$

$$=k(a-k)+\frac{1}{2}k^2(a-k)^2+\cdots+\frac{1}{n}k^n(a-k)^n+\cdots$$

Let us now equate the coefficients of k^n from both sides. On the left-hand side, the coefficient of k^n is $\dfrac{1}{n}\left(x^n+\dfrac{1}{x^n}\right)$ $=\dfrac{2}{n}\cos n\theta$. On the right-hand side, the coefficient of k^n is obtained by collecting the coefficient of k^n from the expansion of $\dfrac{1}{n}k^n(a-k)^n$ and the terms which precede it.

The coefficient of k^n in $\frac{1}{n}k^n(a-k)^n = \frac{a^n}{n}$

$$\frac{1}{n-1}k^{n-1}(a-k)^{n-1}$$

$$= -\frac{1}{n-1}\cdot(n-1)a^{n-2}$$

$$\frac{1}{n-2}k^{n-2}(a-k)^{n-2}$$

$$= \frac{1}{n-2}\cdot\frac{(n-2)(n-3)}{\lfloor 2}a^{n-4}$$

and generally the coefficient of k^n in $\frac{1}{n-r}k^{n-r}(a-k)^{n-r}$ is

$$\frac{(-1)^r}{n-r}\cdot\frac{(n-r)(n-r-1)\cdots(n-2r+1)}{\lfloor r}\cdot a^{n-2r}.$$

$\therefore$ **2 cos nθ**

$$= (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{\lfloor 2}(2\cos\theta)^{n-4} - \cdots$$

Note. The series on the right-hand side will continue so long as the powers of $2\cos\theta$ are not negative. The last term is

$$(-1)^{\frac{n-1}{2}}n(2\cos\theta) \text{ or } (-1)^{\frac{n}{2}}.2$$

according as n is *odd* or *even*.

95. Expansion of sin nθ in *descending* powers of cos θ (*n being a positive integer*).

Let $\quad k = \cos\theta + i\sin\theta$;

then $\quad k - k^{-1} = 2i\sin\theta$ and $k^n - k^{-n} = 2i\sin n\theta$.

Now,

$$\frac{\sin\theta}{1-2x\cos\theta+x^2} = \frac{1}{2i}\cdot\frac{k-k^{-1}}{1-(k+k^{-1})x+x^2}$$

$$= \frac{1}{2ix}\left[\frac{1}{1-xk} - \frac{1}{1-xk^{-1}}\right] = \frac{1}{2ix}\left[(1-xk)^{-1} - (1-xk^{-1})^{-1}\right]$$

$$= \frac{1}{2ix}\left[(1 + xk + x^2k^2 + \cdots) - (1 + xk^{-1} + x^2k^{-2} + \cdots)\right],$$

if $|x| < 1$

$$= \frac{1}{2ix}\left[x(k - k^{-1}) + x^2(k^2 - k^{-2}) + \cdots + x^n(k^n - k^{-n}) + \cdots\right]$$

$$= \sin\theta + x\sin 2\theta + x^2\sin 3\theta + \cdots + x^{n-1}\sin n\theta + \cdots \quad (1)$$

Now, $\dfrac{1}{1 - 2x\cos\theta + x^2} = (1 - 2x\cos\theta + x^2)^{-1}$

$= \{1 - x(a - x)\}^{-1}$, where $a = 2\cos\theta$

$= 1 + x(a - x) + x^2(a - x)^2 + \cdots + x^n(a - x)^n + \cdots$

Therefore, the relation (1) may be written as

$$1 + x(a - x) + x^2(a - x)^2 + \cdots + x^{n-1}(a - x)^{n-1} + x^n(a - x)^n + \cdots$$

$$= \frac{1}{\sin\theta}\left[\sin\theta + x\sin 2\theta + x^2\sin 3\theta + \cdots + x^{n-1}\sin n\theta + \cdots\right] \quad \cdots \quad (2)$$

Let us now equate the coefficients of x^{n-1} from both sides of (2). On the right side, the coeff. of $x^{n-1} = \dfrac{\sin n\theta}{\sin\theta}$.

On the left side, the coeff. of x^{n-1} is obtained by collecting the coeff. of x^{n-1} from the expansion of $x^{n-1}(a - x)^{n-1}$ and of the terms which precede it.

Thus, the coeff. of x^{n-1} in $x^{n-1}(a - x)^{n-1} = a^{n-1}$

$\cdots \quad \cdots \quad x^{n-2}(a - x)^{n-2} = -(n - 2)a^{n-3}$

$\cdots \quad \cdots \quad x^{n-3}(a - x)^{n-3} = \dfrac{(n - 3)(n - 4)}{2!}a^{n-5}$

and generally, the coeff. of x^{n-1} in $x^{n-r}(a - x)^{n-r}$ is

$$\frac{(-1)^{r-1}(n - r)(n - r - 1)\cdots(n - 2r + 2)}{(r - 1)!}a^{(n-2r+1)}.$$

$$\therefore \quad \sin n\theta = \sin\theta\,[(2\cos\theta)^{n-1} - (n-2)(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{2\,!}(2\cos\theta)^{n-5} - \cdots].$$

Note. The series on the right side will continue so long as the powers of $2\cos\theta$ are not negative. The last term is

$$(-1)^{\frac{n-1}{2}} \text{ or } (-1)^{\frac{n}{2}-1}(n\cos\theta)$$

according as n is *odd* or *even*.

96. Expansions of $\sin n\theta$, $\cos n\theta$ in *ascending* powers of $\sin\theta$, $\cos\theta$ *(n being any integer).*

It is evident from Article 94 that when n is an *even positive* integer, $\cos n\theta$ can be arranged in a series of even powers of $\sin\theta$. So we can assume

$$\cos n\theta = 1 + A_2\sin^2\theta + A_4\sin^4\theta + \cdots + A_n\sin^n\theta. \quad \cdots \text{(i)}$$

The constant term is taken as 1, because when $\theta = 0$, $\cos n\theta = 1$ and $\sin\theta = 0$.

Putting $\theta + h$ for θ on both sides of (i), we have,

$$\begin{aligned}\cos n\theta\cos nh - \sin n\theta\sin nh &= 1 + A_2(\sin\theta\cos h + \cos\theta\sin h)^2 \\ &\quad + A_4(\sin\theta\cos h + \cos\theta\sin h)^4 + \cdots\cdots \\ &\quad + A_{2r}(\sin\theta\cos h + \cos\theta\sin h)^{2r} + \cdots\end{aligned}$$

Now, for $\cos nh$, $\sin nh$, $\cos h$, $\sin h$, write the respective series in ascending powers of h. Then

$$\cos n\theta\,(1 - \tfrac{1}{2}n^2h^2 + \cdots) - \sin n\theta\,(nh - \tfrac{1}{6}n^3h^3 + \cdots)$$
$$= 1 + A_2\,[\sin\theta\,(1 - \tfrac{1}{2}h^2 + \cdots) + \cos\theta\,(h - \tfrac{1}{6}h^3 + \cdots)]^2 + \cdots \quad \text{(ii)}$$

Since, this result is true for all values of h, we have, on equating the coefficients of h^2 from both sides,

$$-\tfrac{1}{2}n^2\cos n\theta = A_2(\cos^2\theta - \sin^2\theta) + \cdots$$

Now, to obtain the general relation between the coefficients A_2, A_4......, consider the term

$$A_{2r}(\sin\theta\cos h + \cos\theta\sin h)^{2r},$$

$$\text{or,} \quad A_{2r}\,[\sin\theta\,(1 - \tfrac{1}{2}h^2 + \cdots) + \cos\theta\,(h - \tfrac{1}{6}h^3 + \cdots)]^{2r},$$

or, A_{2r} $[\sin\theta + h\cos\theta - \frac{1}{2}h^2\sin\theta]^{2r}$ (neglecting powers of h higher than h^2).

Now, the coefficient of h^2 in the expansion of this term is

$$A_{2r}\left\{\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta\cos^2\theta - r\sin^{2r}\theta\right\}$$

i.e., $$A_{2r}\left\{\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta(1-\sin^2\theta) - r\sin^{2r}\theta\right\}$$

i.e., $$A_{2r}\left\{\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta - \sin^{2r}\theta\left(r + \frac{2r(2r-1)}{1.2}\right)\right\}.$$

Similarly, the coefficient of h^2 in the expansion of $A_{2r+2}\{\sin(\theta+h)\}^{2r+2}$ is

$$A_{2r+2}\left\{\frac{(2r+2)(2r+1)}{1.2}\sin^{2r}\theta - \sin^{2r+2}\theta \times\left(r+1+\frac{(2r+2)(2r+1)}{1.2}\right)\right\}.$$

Hence, we obtain as the coefficient of $\sin^{2r}\theta$ in the expansion of $-\frac{1}{2}n^2\cos n\theta$ the expression,

$$-A_{2r}\left\{r + \frac{2r(2r-1)}{1.2}\right\} + A_{2r+2}\left\{\frac{(2r+2)(2r+1)}{1.2}\right\}.$$

But by series (i), A_{2r} is the coefficient of $\sin^{2r}\theta$ in the expansion of $\cos n\theta$; hence, $-\frac{1}{2}n^2A_{2r}$ is the coefficient of $\sin^{2r}\theta$ in the expansion of $-\frac{1}{2}n^2\cos n\theta$.

$$\therefore\ -\tfrac{1}{2}n^2A_{2r} = -A_{2r}\left\{\frac{2r(2r-1)}{1.2} + r\right\} + A_{2r+2}\left\{\frac{(2r+2)(2r+1)}{1.2}\right\},$$

or, $$A_{2r+2} = -\frac{n^2-(2r)^2}{(2r+1)(2r+2)}A_{2r}.$$

Putting $r=0, 1, 2, 3, \ldots$, in succession, we obtain,

$$A_2 = -\frac{n^2}{1.2}A_0 = -\frac{r^2}{2\,!}$$

$$A_4 = -\frac{n^2-2^2}{3.4}A_2 = \frac{n^2(n^2-2^2)}{4\,!}$$

and so on.

Thus, when n is even,

$$\cos n\theta = 1 - \frac{n^2}{2!}\sin^2\theta + \frac{n^2(n^2-2^2)}{4!}\sin^4\theta - \cdots \quad (1)$$

Equating the coefficients of h from both sides of (ii), we shall obtain,

$$-n \sin n\theta = 2A_2 \sin\theta\cos\theta + 4A_4 \sin^3\theta\cos\theta + \cdots$$
$$+ 2rA_{2r}\sin^{2r-1}\theta\cos\theta + \cdots$$

Now, substituting the values of A_2, A_4, etc., we have,

when n is even,

$$\sin n\theta = n\cos\theta\left\{\sin\theta - \frac{n^2-2^2}{3!}\sin^3\theta + \frac{(n^2-2^2)(n^2-4^2)}{5!}\sin^5\theta - \cdots\right\} \quad \cdots \quad (2)$$

When n is an *odd integer*, we can assume

$$\sin n\theta = A_1 \sin\theta + A_3 \sin^3\theta + A_5 \sin^5\theta + \cdots + A_n \sin^n\theta.$$

Since, $\sin(-n\theta) = -\sin n\theta$, the expansion of $\sin n\theta$ would not contain even powers of $\sin\theta$; since, $\sin 0 = 0$, there will be no constant term.

Now, proceeding as before, we shall find,

when n is odd,

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \cdots \quad (3)$$

$$\cos n\theta = \cos\theta\left\{1 - \frac{n^2-1^2}{2!}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4\theta - \cdots\right\} \quad (4)$$

Note. Since n is a positive integer, the four series obtained above are all finite.

Changing θ into $\frac{\pi}{2}-\theta$ in the formulæ (1), (2), (3), (4), we have, if n be an *even integer*,

$$(-1)^{\frac{n}{2}}\cos n\theta=1-\frac{n^2}{2\,!}\cos^2\theta+\frac{n^2(n^2-2^2)}{4\,!}\cos^4\theta-\cdots \quad (5)$$

$$(-1)^{\frac{n}{2}+1}\sin n\theta=n\sin\theta\left\{\cos\theta-\frac{n^2-2^2}{3\,!}\cos^3\theta\right.$$
$$\left.+\frac{(n^2-2^2)(n^2-4^2)}{5\,!}\cos^5\theta-\cdots\right\}. \quad \cdots \quad (6)$$

If n be an *odd integer*,

$$(-1)^{\frac{n-1}{2}}\cos n\theta=n\cos\theta-\frac{n(n^2-1^2)}{3\,!}\cos^3\theta$$
$$+\frac{n(n^2-1^2)(n^2-3^2)}{5\,!}\cos^5\theta-\cdots \quad \cdots \quad (7)$$

$$(-1)^{\frac{n-1}{2}}\sin n\theta=\sin\theta\left\{1-\frac{n^2-1^2}{2\,!}\cos^2\theta\right.$$
$$\left.+\frac{(n^2-1^2)(n^2-3^2)}{4\,!}\cos^4\theta-\cdots\right\}. \quad \cdots \quad (8)$$

97. Expansions of sin nθ and cos nθ *(n being not an integer).*

$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n=\cos^n\theta\,(1+i\tan\theta)^n.$$

If $\tan\theta$ be numerically less than unity, $\cos^n\theta(1+i\tan\theta)^n$ can be expanded by the Binomial theorem and since $\cos^r\theta=(1-\sin^2\theta)^{\frac{r}{2}}$, the expansion can be expressed in a series of sines.

Again, since $\cos(-n\theta)=\cos n\theta$, the series for $\cos n\theta$ will contain only even powers of $\sin\theta$; and since $\sin(-n\theta)=-\sin n\theta$, the series for $\sin n\theta$ will contain only odd powers of $\sin\theta$.

Also, since $\cos 0 = 1$ and $\sin 0 = 0$, the constant term in the series for $\cos n\theta$ will be 1 and there will be no constant term in the series for $\sin n\theta$.

Hence, we can assume, as in the previous article,

$$\cos n\theta = 1 + A_2 \sin^2\theta + A_4 \sin^4\theta + \cdots$$

$$\sin n\theta = A_1 \sin\theta + A_3 \sin^3\theta + \cdots$$

and proceeding exactly in the same way, the four series (1), (2), (3), (4) of the previous article can be obtained. In this case θ shall lie between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$.

The demonstration given above establishes the four series under the restriction that $\tan\theta$ must be numerically less than unity, *i.e.*, θ must lie between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$. It can, however, be shown that the series will hold even when θ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.*

Ex. 3. *Prove that*

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \cdots \qquad [\textit{C. H. 1921.}]$$

For *any general value of n*, we have

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \cdots$$

Since, $\sin n\theta = n\theta - \dfrac{n^3\theta^3}{3!} + \cdots$

we have by equating the coefficients of n on both sides,

$$\theta = \sin\theta + \frac{1^2}{3!}\sin^3\theta + \frac{1^2.3^2}{5!}\sin^5\theta + \cdots$$

$$= \sin\theta + \frac{1}{2}\cdot\frac{\sin^3\theta}{3} + \frac{1.3}{2.4}\cdot\frac{\sin^5\theta}{5} + \cdots$$

Let us put $\sin\theta = x$, so that $\theta = \sin^{-1}x$.

$$\therefore\quad \sin^{-1}x = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \cdots$$

*See Hobson's Trigonometry.

Ex. 4. *If* $\tan\theta = \dfrac{a \sin cx}{1-a\cos cx}$ *and* $r^2 = 1-2a\cos cx + a^2$, *show that*

$$1-\frac{n}{1!}a\cos cx+\frac{n(n-1)}{2!}a^2\cos 2cx-\cdots+(-1)^n a^n\cos ncx=r^n\cos n\theta.$$

[*C. H. 1925, '29.*]

Put for each cosine its exponential value ; then the proposed series

$$=\tfrac{1}{2}(1-ae^{icx})^n+\tfrac{1}{2}(1-ae^{-icx})^n$$

$$=\tfrac{1}{2}(1-a\cos cx-ia\sin cx)^n+\tfrac{1}{2}(1-a\cos cx+ia\sin cx)^n\cdots(1)$$

Now, putting $1-a\cos cx=r\cos\theta$, and $a\sin cx=r\sin\theta$,

so that $\tan\theta=\dfrac{a\sin cx}{1-a\cos cx}$ and $r^2=1-2a\cos cx+a^2$, (1) becomes

$$=\tfrac{1}{2}(r\cos\theta-i\,r\sin\theta)^n+\tfrac{1}{2}(r\cos\theta+i\,r\sin\theta)^n$$

$$=\tfrac{1}{2}r^n(\cos n\theta-i\sin n\theta)+\tfrac{1}{2}r^n(\cos n\theta+i\sin n\theta)$$

$$=r^n\cos n\theta.$$

Ex. 5. *Expand* $\dfrac{2x\cos\theta}{1-2x\sin\theta+x^2}$, $(\,|\,x\,|<1\,)$

in a series of sines and cosines of multiples of θ. [*C. H. 1951.*]

Let $z=\cos\theta+i\sin\theta$; then $z^n-z^{-n}=2i\sin n\theta$. [see *Art. 90*]

$$\therefore\quad i(z^n-z^{-n})=-2\sin n\theta.$$

Also, $i(z-z^{-1})=-2\sin\theta$ and $z+z^{-1}=2\cos\theta$.

Thus, $$\frac{2x\cos\theta}{1-2x\sin\theta+x^2}=\frac{x(z+z^{-1})}{1+ix(z-z^{-1})-i^2x^2}=\frac{x(z+z^{-1})}{(1+izx)(1-iz^{-1}x)}$$

$$=\frac{i}{1+izx}-\frac{i}{1-iz^{-1}x}=i(1+izx)^{-1}-i(1-iz^{-1}x)^{-1}$$

$$=i[1-izx+i^2z^2x^2-i^3z^3x^3+\cdots]$$

$$-i[1+iz^{-1}x+i^2z^{-2}x^2+i^3z^{-3}x^3+\cdots]$$

$$=(z+z^{-1})x-i(z^2-z^{-2})x^2-(z^3+z^{-3})x^3+i(z^4-z^{-4})x^4+\cdots$$

$$=2x\cos\theta+2x^2\sin 2\theta-2x^3\cos 3\theta-2x^4\sin 4\theta+\cdots$$

EXAMPLES XIII

1. Prove that

(i) $\cos^8 x = \frac{1}{128}[\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35]$.

(ii) $\sin^9 x = \frac{1}{256}[\sin 9x - 9\sin 7x + 36\sin 5x - 84\sin 3x + 126\sin x]$.

(iii) $2^8 \sin^5\theta \cos^4\theta = \sin 9\theta - \sin 7\theta - 4\sin 5\theta + 4\sin 3\theta + 6\sin\theta$.

(iv) $\sin^4\theta\cos^3\theta = \frac{1}{2^6}[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta]$.

2. Expand $\cos^n 3\theta$ in terms of cosines of multiples of θ, n being a positive integer.

3. Prove that

(i) $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

(ii) $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$.

(iii) $1 + \cos 10\theta = 2[16\cos^5\theta - 20\cos^3\theta + 5\cos\theta]^2$.

(iv) $\sin 8\theta = \sin\theta\,[128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta]$.

4. Prove that

(i) $\frac{1}{2}\theta^2 = \frac{1}{2}\sin^2\theta + \frac{2}{3}\cdot\frac{\sin^4\theta}{4} + \frac{2.4}{3.5}\cdot\frac{\sin^6\theta}{6} + \cdots$

[*C. H. 1934.*]

(ii) $\frac{1}{6}\theta^3 = \frac{1}{3!}\sin^3\theta + \frac{1^2+3^2}{5!}\sin^5\theta + \frac{1^2.3^2+1^2.5^2+3^2.5^2}{7!}\sin^7\theta + \cdots$

5. Show that

(i) $(\sin^{-1}x)^2 = x^2 + \frac{2}{3}\cdot\frac{x^4}{2} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{x^6}{3} + \cdots$ [*C. H. 1932.*]

[*Put* $\sin\theta = x$ *in Ex.* 4(i).]

(ii) $\frac{1}{6}(\sin^{-1}x)^3 = \frac{1}{2}\cdot\frac{x^3}{3} + \left(\frac{1}{1^2}+\frac{1}{3^2}\right)\cdot\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\cdots$

[*Put* $\sin\theta = x$ *in Ex.* 4(*ii*).]

6. Prove that

(i) $\theta\sec\theta = \sin\theta + \frac{2}{3}\sin^3\theta + \frac{2.4}{3.5}\sin^5\theta + \cdots$

(ii) $\sec\theta = 1 + \frac{1}{2}\sin^2\theta + \frac{1.3}{2.4}\sin^4\theta + \cdots$ [*C. H. 1933.*]

7. Prove that

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 + \cdots$$

[*Put* $\sin\theta = x$ *in Ex.* 6(*i*).]

8. Prove that

$$\tan^{-1}y = \frac{y}{1+y^2}\left\{1 + \frac{2}{3}\cdot\frac{y^2}{1+y^2} + \frac{2.4}{3.5}\left(\frac{y^2}{1+y^2}\right)^2 + \cdots\right\}.$$

[*Put* $\theta = \tan^{-1}y$ *in Ex.* 6(*i*).]

9. If $|x|$ be < 1, prove that

(i) $\frac{\sin\theta}{1-2x\cos\theta+x^2} = \sin\theta + x\sin 2\theta + x^2\sin 3\theta + \cdots$

[*C. H. 1940.*]

(ii) $\frac{1-x^2}{1-2x\cos\theta+x^2} = 1 + 2x\cos\theta + 2x^2\cos 2\theta$

$+ 2x^3\cos 3\theta + \cdots$

[*C. H. 1946.*]

(iii) $\frac{\cos\theta - x}{1-2x\cos\theta+x^2} = \cos\theta + x\cos 2\theta + x^2\cos 3\theta + \ldots$

10. Prove that $\pi^2 = 18\sum_{1}^{\infty}\frac{\lfloor n \lfloor n}{\lfloor 2n+2}$ [*C. H. 1935.*]

[*Put* $\theta = \frac{1}{2}\pi$ *in Ex.* 4(*i*).]

11. Establish the identity $\cos^n\theta\,[\cos n\theta + i\sin n\theta] = (1 - i\tan\theta)^{-n}$ and hence deduce the expansions of $\cos^n\theta\cos n\theta$ and $\cos^n\theta\sin n\theta$ in powers of $\tan\theta$.

12. Establish the formulæ

$$\cos 5\theta = \cos\theta\,(1 - 12\sin^2\theta + 15\sin^4\theta)$$

$$\sin 5\theta = \sin\theta\,(1 - 12\cos^2\theta + 15\cos^4\theta).$$

Reduce the quantity

$V = (\sin\theta + \cos\theta)(1 + 4\sin\theta\cos\theta - 16\sin^2\theta\cos^2\theta)$ in the form $A\cos 5\theta + B\sin 5\theta$, where A and B are certain numerical constants. Hence, find an expansion of V in ascending powers of θ as far as the third significant term. [*C. H. 1927.*]

13. Expand $(1 - 2x\cos\theta + x^2)^{-1}$ in powers of x, $|x| < 1$.

14. Expand $\tan^{-1}\dfrac{x\sin\theta}{1 - x\cos\theta}$ in powers of x, $|x| < 1$.

15. Expand $\log(1 - 2x\cos\theta + x^2)$ in a series of cosines of multiples of θ, $|x| < 1$. [*C. H. 1948.*]

16. Prove that $\dfrac{1}{1 + \sin 2\alpha\cos\theta}$ can be expanded in the infinite series

$$\sec 2\alpha[1 - 2\tan\alpha\cos\theta + 2\tan^2\alpha\cos 2\theta - \cdots].$$

[*Given exp.* $= \dfrac{1+t^2}{1-t^2}\cdot\dfrac{1-t^2}{1-2t\cos\theta+t^2}$, *where* $t = -\tan\alpha$.
Now apply Ex. 9(ii).]

17. Prove that

$$\frac{\cos x}{(n-1)!\,(n+1)!} + \frac{\cos 2x}{(n-2)!\,(n+2)!} + \cdots + \frac{\cos nx}{(2n)!}$$
$$= \frac{2^{n-1}(1+\cos x)^n}{2(n!)} - \frac{1}{2(n!)^2}.$$

18. If $\cot\theta = \dfrac{1}{x} + \cot\alpha$, show that

$$\theta = \frac{x}{\sin\alpha}\cdot\sin\alpha - \frac{1}{2}\cdot\frac{x^2}{\sin^2\alpha}\cdot\sin 2\alpha + \frac{1}{3}\cdot\frac{x^3}{\sin^3\alpha}$$

19. Prove that

$$2\sin^2 n\theta = \frac{n^2}{2!}(2s)^2 - \frac{n^2(n^2-1^2)}{4!}(2s)^4 + \frac{n^2(n^2-1^2)(n^2-2^2)}{6!}(2s)^6 - \cdots$$

where $s = \sin\theta$ and n is a positive integer.

20. From the expansion of $\sin^n\theta$ in multiple angles, show that $n^r - n(n-2)^r + \dfrac{n(n-1)}{1.2}(n-4)^r - \cdots$ to $\dfrac{n+1}{2}$ terms $=0$, where n and r are odd positive integers and $r < n$.

[*C. H. 1939.*]

21. If $\tan(\frac{1}{4}\alpha - \theta) = \tan^3\frac{1}{4}\alpha$, show that

$$\theta = \frac{1}{1.3}\sin\alpha - \frac{1}{2.3^2}\sin 2\alpha + \frac{1}{3.3^3}\sin 3\alpha - \cdots$$

[*C. H. 1941.*]

22. Expand $\dfrac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2}$ in the form $\displaystyle\sum_{n=-\infty}^{n=\infty} c_n x^n$ where c_n is a trigonometrical function of θ to be determined ($|x| < 1$).

[*C. H. 1943*]

ANSWERS

2. Put 3θ for θ in the result (2) of Art. 91.

11. $\cos^n\theta\cos n\theta = 1 - \dfrac{n(n+1)}{2!}\tan^2\theta + \dfrac{n(n+1)(n+2)(n+3)}{4!}\tan^4\theta - \cdots$

$\cos^n\theta\sin n\theta = n\tan\theta - \dfrac{n(n+1)(n+2)}{3!}\tan^3\theta + \cdots$

12. $1 + 5\theta - \dfrac{25\theta^2}{2} - \cdots\cdots$ **13.** $1 + x\dfrac{\sin 2\theta}{\sin\theta} + x^2\dfrac{\sin 3\theta}{\sin\theta} + x^3\dfrac{\sin 4\theta}{\sin\theta} + \cdots$

14. $x\sin\theta + \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta + \cdots$

15. $-2[x\cos\theta + \frac{1}{2}x^2\cos 2\theta + \frac{1}{3}x^3\cos 3\theta + \cdots]$.

CHAPTER XIV

RESOLUTION INTO FACTORS

98. Resolution of sin θ into factors.

The proof depends upon successive application of the formulæ

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} = 2 \sin \frac{A}{2} \sin \frac{\pi + A}{2}.$$

Thus, we have,

$$\sin \theta = 2 \sin \frac{\theta}{2} \sin \frac{\pi + \theta}{2}. \qquad \cdots \quad (1)$$

Putting $\frac{\theta}{2}, \frac{\pi+\theta}{2}$ for A successively in the above formula, we get

$$\sin \frac{\theta}{2} = 2 \sin \frac{\theta}{2^2} \cdot \sin \frac{2\pi + \theta}{2^2}$$

$$\frac{\pi + \theta}{2} = 2 \sin \frac{\pi + \theta}{2^2} \sin \frac{3\pi + \theta}{2^2}.$$

Substituting these values in right side of (1) and rearranging, we have

$$\sin \theta = 2^3 \sin \frac{\theta}{2^2} \sin \frac{\pi + \theta}{2^2} \sin \frac{2\pi + \theta}{2^2} \sin \frac{3\pi + \theta}{2^2}.$$

Applying successively the above formula, we get

$$\sin \theta = 2^7 \sin \frac{\theta}{2^3} \sin \frac{\pi + \theta}{2^3} \cdots\cdots \sin \frac{7\pi + \theta}{2^3}$$

$$= 2^{k-1} \sin \frac{\theta}{k} \sin \frac{\pi + \theta}{k} \sin \frac{2\pi + \theta}{k} \cdots \sin \frac{(k-1)\pi + \theta}{k} \quad \cdots \quad (2)$$

where $k = 2^n$.

The last factor in (2), $\sin \dfrac{(k-1)\pi+\theta}{k}$

$$=\sin\left(\pi-\frac{\pi-\theta}{k}\right)=\sin\frac{\pi-\theta}{k}.$$

The last but one factor, $\sin \dfrac{(k-2)\pi+\theta}{k}$

$$=\sin\left(\pi-\frac{2\pi-\theta}{k}\right)=\sin\frac{2\pi-\theta}{k};$$

and so on.

The $(\frac{1}{2}k+1)$th factor from the beginning

$$=\sin\frac{\frac{1}{2}k\pi+\theta}{k}=\sin\left(\frac{\pi}{2}+\frac{\theta}{k}\right)=\cos\frac{\theta}{k}$$

Hence, combining the second and last factors, the third and last but one, and so on, and leaving alone the first and $(\frac{1}{2}k+1)$th factors which have no conjugates, the equation (2) becomes

$$\sin\theta=2^{k-1}\sin\frac{\theta}{k}\left\{\sin^2\frac{\pi}{k}-\sin^2\frac{\theta}{k}\right\}\left\{\sin^2\frac{2\pi}{k}-\sin^2\frac{\theta}{k}\right\}$$
$$\cdots\left\{\sin^2\frac{(\frac{1}{2}k-1)\pi}{k}-\sin^2\frac{\theta}{k}\right\}\cos\frac{\theta}{k}. \qquad \cdots \quad (3)$$

Divide both sides of (3) by $\sin\dfrac{\theta}{k}$ and let θ approach zero; then, since

$$\underset{\theta\to 0}{Lt}\frac{\sin\theta}{\theta}=1 \text{ and } \underset{\theta\to 0}{Lt}\sin^2\frac{\theta}{k}=0,$$

and $$\underset{\theta\to 0}{Lt}\frac{\sin\theta}{\sin\frac{\theta}{k}}=\underset{\theta\to 0}{Lt}\left[k\frac{\sin\theta}{\theta}\cdot\frac{\frac{\theta}{k}}{\sin\frac{\theta}{k}}\right]=k,$$

we have

$$k = 2^{k-1} \sin^2 \frac{\pi}{k} \sin^2 \frac{2\pi}{k} \sin^2 \frac{3\pi}{k} \cdots \sin^2 \frac{(\frac{1}{2}k-1)\pi}{k} \tag{4}$$

Dividing (3) by (4),

$$\sin\theta = k \sin\frac{\theta}{k} \cos\frac{\theta}{k} \left\{1 - \frac{\sin^2 \frac{\theta}{k}}{\sin^2 \frac{\pi}{k}}\right\}\left\{1 - \frac{\sin^2 \frac{\theta}{k}}{\sin^2 \frac{2\pi}{k}}\right\}$$

Now, make k indefinitely large.

Since, $\underset{k\to\infty}{Lt}\left[k\sin\frac{\theta}{k}\right] = \underset{k\to\infty}{Lt}\left[\frac{\sin\frac{\theta}{k}}{\frac{\theta}{k}}\cdot\theta\right] = \theta,$

$$\underset{k\to\infty}{Lt}\left[\frac{\sin^2\frac{\theta}{k}}{\sin^2\frac{\pi}{k}}\right] = \underset{k\to\infty}{Lt}\left[\frac{\sin^2\frac{\theta}{k}}{\frac{\theta^2}{k^2}}\cdot\frac{\frac{\pi^2}{k^2}}{\sin^2\frac{\pi}{k}}\cdot\frac{\theta^2}{\pi^2}\right],$$

and so on, and $\underset{k\to\infty}{Lt} \cos\frac{\theta}{k} = 1$, we have

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \text{ ...... ad. inf.}$$

i.e., $$= \theta \prod_{r=1}^{r=\infty}\left\{1 - \frac{\theta^2}{r^2\pi^2}\right\}.$$

99. Resolution of $\cos\theta$ into factors.

Proceeding as in the previous article, we get

$$\sin\theta = 2^{k-1}\sin\frac{\theta}{k}\sin\frac{\pi+\theta}{k}\sin\frac{2\pi+\theta}{k}\cdots\sin\frac{(k-1)\pi+\theta}{k}. \tag{1}$$

Putting $\frac{1}{2}\pi + \theta$ for θ in the above equation, we get

$$\cos\theta = 2^{k-1}\sin\frac{\pi+2\theta}{2k}\sin\frac{3\pi+2\theta}{2k}\cdots\sin\frac{(2k-1)\pi+2\theta}{2k}.$$

The last factor $= \sin\left\{\pi - \frac{\pi - 2\theta}{2k}\right\} = \sin\frac{\pi - 2\theta}{2k}$,

the last but one factor

$$= \sin\left\{\frac{(2k-3)\pi + 2\theta}{2k}\right\} = \sin\frac{3\pi - 2\theta}{2k}, \text{ and so on.}$$

Now, combining the factors in pairs, the first and the last, the second and the last but one, and so on, we get

$$\cos\theta = 2^{k-1}\left\{\sin\frac{\pi + 2\theta}{2k}\sin\frac{\pi - 2\theta}{2k}\right\}$$

$$\times\left\{\sin\frac{3\pi + 2\theta}{2k}\sin\frac{3\pi - 2\theta}{2k}\right\}\cdots\cdots$$

$$= 2^{k-1}\left\{\sin^2\frac{\pi}{2k} - \sin^2\frac{2\theta}{2k}\right\}\left\{\sin^2\frac{3\pi}{2k} - \sin^2\frac{2\theta}{2k}\right\} \quad \cdots \quad (2)$$

In (2) letting θ approach zero, we get

$$1 = 2^{k-1}\sin^2\frac{\pi}{2k}\sin^2\frac{3\pi}{2k}\sin^2\frac{5\pi}{2k}\cdots\cdots \quad \cdots \quad (3)$$

Dividing (2) by (3) and making k indefinitely large, we have as in the last article

$$\cos\theta = \left[1 - \frac{4\theta^2}{\pi^2}\right]\left[1 - \frac{4\theta^2}{3^2\pi^2}\right]\left[1 - \frac{4\theta^2}{5^2\pi^2}\right]\cdots\cdots \textbf{ ad. inf.}$$

$$i.e.,\quad = \prod_{r=1}^{r=\infty}\left[1 - \frac{4\theta^2}{(2r-1)^2\pi^2}\right].$$

Obs. Since, $\cos\theta = \frac{\sin 2\theta}{2\sin\theta}$, $\cos\theta$ may be resolved into factors by means of the factors of $\sin 2\theta$ and $\sin\theta$.

100. Sum of powers of the reciprocals of natural numbers.

From Art. 54, we have

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

and from Art. 98, we have

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdots$$

Equating the above two values of $\sin\theta$, we get the identity

$$1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\cdots=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\quad\cdots\quad(1)$$

Similarly, equating the two values of $\cos\theta$, as given in Arts. 53 and 99, we get the identity

$$1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\quad\cdots\quad(2)$$

Taking logarithms of both sides of (1), we have

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\cdots$$
$$=\log\left\{1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)\right\}.$$

Expanding each of the logarithms in the above identity,

$$\left\{\frac{\theta^2}{\pi^2}+\frac{1}{2}\cdot\frac{\theta^4}{\pi^4}+\cdots\right\}+\left\{\frac{\theta^2}{2^2\pi^2}+\frac{1}{2}\cdot\frac{\theta^4}{2^4\pi^4}+\cdots\right\}$$
$$+\left\{\frac{\theta^2}{3^2\pi^2}+\frac{1}{2}\cdot\frac{\theta^4}{3^4\pi^4}+\cdots\right\}+\cdots$$
$$=\left\{\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right\}+\frac{1}{2}\left\{\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right\}^2+\cdots\cdots\quad(3)$$

Equating the coefficients of the various powers of θ^2 from both sides in this identity, we get the sums of the various powers of the reciprocals of natural numbers.

Thus, equating the coefficients of θ^2 and θ^4, we have

$$\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=\frac{1}{6},$$

and $$\frac{1}{2}\cdot\frac{1}{\pi^4}\left\{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right\}=-\frac{1}{120}+\frac{1}{72}.$$

Hence, $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots=\frac{\pi^2}{6} \quad \cdots \quad (4)$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\cdots=\frac{\pi^4}{90} \quad \cdots \quad (5)$$

Similarly, taking logarithms of both sides of (2) and expanding each of these logarithms and equating the coefficients of θ^2 and θ^4, we have

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8} \quad \cdots \quad (6)$$

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\cdots=\frac{\pi^4}{96} \quad \cdots \quad (7)$$

101. Factors of sinh x and cosh x.

Putting $\theta=ix$ in the factors for $\sin\theta$, we get

$$\sin ix=ix\left[1-\frac{i^2x^2}{\pi^2}\right]\left[1-\frac{i^2x^2}{2^2\pi^2}\right]\left[1-\frac{i^2x^2}{3^2\pi^2}\right]\cdots\cdots$$

$$=ix\left[1+\frac{x^2}{\pi^2}\right]\left[1+\frac{x^2}{2^2\pi^2}\right]\left[1+\frac{x^2}{3^2\pi^2}\right]\cdots\cdots$$

Now, $\sin ix=i\sinh x$.

$$\therefore \quad \sinh x=x\left[1+\frac{x^2}{\pi^2}\right]\left[1+\frac{x^2}{2^2\pi^2}\right]\left[1+\frac{x^2}{3^2\pi^2}\right]\cdots\cdots$$

$$=x\prod_1^\infty\left[1+\frac{x^2}{r^2\pi^2}\right].$$

Similarly, putting $\theta=ix$ in the factors for $\cos\theta$ and noting that $\cos ix=\cosh x$, we get

$$\cosh x=\left[1+\frac{4x^2}{\pi^2}\right]\left[1+\frac{4x^2}{3^2\pi^2}\right]\left[1+\frac{4x^2}{5^2\pi^2}\right]\cdots\cdots$$

$$=\prod_1^\infty\left[1+\frac{4x^2}{(2r-1)^2\pi^2}\right].$$

102. Ex. 1. *Prove that*

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots \textit{ad. inf.} \qquad [\textit{C. H. 1950.}]$$

In the expression for $\sin\theta$ in factors, put $\theta=\frac{\pi}{2}$; then

$$1=\frac{\pi}{2}\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{4^2}\right]\left[1-\frac{1}{6^2}\right]\cdots\cdots$$

$$=\frac{\pi}{2}\cdot\frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdots\cdots$$

$$\therefore \quad \frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdots$$

Note. This is known as *Walli's theorem.* It is also stated thus : When n is very large,

$$\sqrt{\tfrac{1}{2}\pi\,(2n+1)}=\frac{2.4.6\cdots 2n}{1.3.5\cdots(2n-1)} \text{ approximately.}$$

When n is very large,

$$\frac{2}{\pi}=\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{4^2}\right]\cdots\left[1-\frac{1}{(2n)^2}\right] \text{ approximately}$$

$$\therefore \quad \frac{2}{\pi}=\frac{1^2.3^2.5^2\cdots(2n-1)^2(2n+1)}{2^2.4^2.6^2\cdots(2n)^2}$$

$$\therefore \quad \sqrt{\tfrac{1}{2}\pi(2n+1)}=\frac{2.4.6\cdots 2n}{1.3.5\cdots(2n-1)}.$$

It is also put in the form $\left[\text{remembering that } \underset{n\to\infty}{Lt}\sqrt{\frac{2n}{2n+1}}=1\right]$

$$\sqrt{\pi}=\underset{n\to\infty}{Lt}\ \frac{1}{\sqrt{n}}\cdot\frac{2.4.6\cdots 2n}{1.3.5\cdots(2n-1)}.$$

Ex. 2. *Prove that*

$$\cot x=\frac{1}{x}-\frac{2x}{\pi^2-x^2}-\frac{2x}{2^2\pi^2-x^2}-\cdots\cdots$$

[*C. H. 1942, '44, '47, '57.*]

Since, $\sin x=x\prod_{1}^{\infty}\left[1-\frac{x^2}{r^2\pi^2}\right],$

we have, when x is not a multiple of π,

$$\log\sin x=\log x+\Sigma\log\left(1-\frac{x^2}{r^2\pi^2}\right). \qquad \cdots \qquad (1)$$

$$\therefore\quad \log\sin(x+h)=\log(x+h)+\Sigma\log\left\{1-\frac{(x+h)^2}{r^2\pi^2}\right\}. \qquad \cdots \quad (2)$$

Subtracting (1) from (2),

$$\log\frac{\sin(x+h)}{\sin x}=\log\frac{x+h}{x}+\Sigma\log\left\{\frac{r^2\pi^2-(x+h)^2}{r^2\pi^2-x^2}\right\}. \qquad \cdots \quad (3)$$

$$\text{Left side of (3)}=\log\frac{\sin x\cos h+\cos x\sin h}{\sin x}$$

$$=\log(\cos h+\cot x\sin h)$$

$$=\log\{1+h\cot x+\text{higher powers of } h\}$$

[by writing down the expansions of $\sin h$ and $\cos h$ in powers of h]

$$=h\cot x+\text{higher powers of } h.$$

$$\text{Right side of (3)}=\log\left[1+\frac{h}{x}\right]+\Sigma\log\left\{1-\frac{2x}{r^2\pi^2-x^2}\,h-\frac{h^2}{r^2\pi^2-x^2}\right\}$$

$$=h\cdot\frac{1}{x}-h\,\Sigma\,\frac{2x}{r^2\pi^2-x^2}+\text{higher powers of } h.$$

$\therefore$ equating coefficients of h from both sides of (3), we get

$$\cot x=\frac{1}{x}-\Sigma\,\frac{2x}{r^2\pi^2-x^2}.$$

Note 1. The above results may be much more easily obtained by *differentiating* both sides of (1) with respect to x. Students acquianted with the methods of Calculus may easily see that replacing x by $x+h$ in a function $f(x)$ and then equating coefficients of h is equivalent to differentiating $f(x)$ with respect to x.

Note 2. $\tan x$ can also be expanded from $\log\cos x$ by a similar method, or by differentiation.

Note 3. For another method, see Ex. XIV(a), sum no. 7(ii).

Ex. 3. *Prove that* $\cos 2x+\cos 2y$

$$=2\cos^2 x\prod_1^\infty\left[\left\{1-\frac{4y^2}{\{(2r-1)\pi+2x\}^2}\right\}\left\{1-\frac{4y^2}{\{(2r-1)\pi-2x\}^2}\right\}\right].$$

$$\cos 2x+\cos 2y=2\cos(x+y)\cos(x-y) \qquad \cdots \qquad \cdots \quad (1)$$

$$=2\cdot\prod_1^\infty\left[1-\frac{4(x+y)^2}{(2r-1)^2\pi^2}\right]\prod_1^\infty\left[1-\frac{4(x-y)^2}{(2r-1)^2\pi^2}\right]$$

$$1-\frac{4(x+y)^2}{(2r-1)^2\pi^2}=\frac{\{(2r-1)\pi+2x+2y\}\{(2r-1)\pi-2x-2y\}}{(2r-1)^2\pi^2} \qquad \cdots \quad (2)$$

$$1-\frac{4(x-y)^2}{(2r-1)^2\pi^2}=\frac{\{(2r-1)\pi+2x-2y\}\{(2r-1)\pi-2x+2y\}}{(2r-1)^2\pi^2} \quad \cdots \quad (3)$$

Now, combining the 1st factor of (2) with the 1st factor of (3) and 2nd factor of (2) with the 2nd factor of (3), we have

$\cos 2x+\cos 2y$

$$=2\prod_1^\infty \frac{[\{(2r-1)\pi+2x\}^2-4y^2][\{(2r-1)\pi-2x\}^2-4y^2]}{(2r-1)^4\pi^4}. \quad (4)$$

Putting $y=0$,

$$1+\cos 2x=2\prod_1^\infty \frac{\{(2r-1)\pi+2x\}^2\{(2r-1)\pi-2x\}^2}{(2r-1)^4\pi^4}. \quad \cdots \quad (5)$$

Dividing (4) by (5), and noting that $1+\cos 2x=2\cos^2 x$, we get the required result.

EXAMPLES XIV(a)

1. Prove that

(i) $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\cdots\cdots=\dfrac{\pi^2}{12}$. [*C. H. 1951.*]

$\left[\textit{Left side}=\Sigma\dfrac{1}{n^2}-2\Sigma\dfrac{1}{(2n)^2}=\Sigma\dfrac{1}{n^2}-\dfrac{1}{2}\Sigma\dfrac{1}{n^2}\cdot \right]$

(ii) $\dfrac{1}{1^6}+\dfrac{1}{2^6}+\dfrac{1}{3^6}+\cdots\cdots=\dfrac{\pi^6}{945}$.

[*Equate coeff. of θ^6 in the identity (3) of Art. 100.*]

(iii) $\dfrac{2}{1^4}+\dfrac{5}{2^4}+\dfrac{10}{3^4}+\dfrac{17}{4^4}+\cdots=\dfrac{\pi^2}{6}+\dfrac{\pi^4}{90}$.

$\left[\textit{nth term}=\dfrac{n^2+1}{n^4}=\dfrac{1}{n^2}+\dfrac{1}{n^4}\cdot \right]$

(iv) $\dfrac{1}{3^4}+\dfrac{3}{5^4}+\dfrac{6}{7^4}+\dfrac{10}{9^4}+\cdots=\dfrac{\pi^2}{64}\left(1-\dfrac{\pi^2}{12}\right)$.

$\left[\textit{nth term}=\dfrac{1}{8}\dfrac{(2n+1)^2-1}{(2n+1)^4}=\dfrac{1}{8}\left\{\dfrac{1}{(2n+1)^2}-\dfrac{1}{(2n+1)^4}\right\}\right]$

(v) $\dfrac{1}{1.2}+\dfrac{1}{2.4}+\dfrac{1}{3.6}+\dfrac{1}{4.8}+\cdots\cdots\cdots=\dfrac{\pi^2}{12}.$

(vi) $\left(\dfrac{1}{1.2.3}\right)^2+\left(\dfrac{1}{2.3.4}\right)^2+\left(\dfrac{1}{3.4.5}\right)^2+\cdots=\dfrac{\pi^2}{4}-\dfrac{39}{16}.$

2. Show that

(i) $\displaystyle\sum_{1}^{\infty}\frac{1}{n^2\,(n+1)^2}=\frac{\pi^2}{3}-3.$

(ii) $\displaystyle\sum_{1}^{\infty}\frac{1}{n^3\,(n+1)^3}=10-\pi^2.$

3. Prove that the sum of the products taken two at a time of the squares of the reciprocals of

(i) all positive integers is $\dfrac{\pi^4}{120}$.

(ii) all positive odd integers is $\dfrac{\pi^4}{384}$. [*C. H. 1936.*]

4. Show that

(i) $\dfrac{\pi}{3}=\dfrac{1^2.6^2}{5.7}.\dfrac{2^2.6^2}{11.13}.\dfrac{3^2.6^2}{17.19}.\dfrac{4^2.6^2}{23.25}\cdots\cdots$

[*Put* $\frac{1}{6}\pi$ *for* θ *in the factors for* $\sin\theta$.]

(ii) $\sqrt{2}=\dfrac{4}{3}.\dfrac{36}{35}.\dfrac{100}{99}.\dfrac{196}{195}\cdots\cdots$

[*Put* $\frac{1}{4}\pi$ *for* θ *in the factors for* $\cos\theta$.]

5. If 2, 3, 5,...... are all the prime numbers, show that

(i) $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{5^2}\right)\cdots\cdots=\dfrac{6}{\pi^2}.$

(ii) $\left(1+\dfrac{1}{2^2}\right)\left(1+\dfrac{1}{3^2}\right)\left(1+\dfrac{1}{5^2}\right)\cdots\cdots=\dfrac{15}{\pi^2}.$

(iii) $\dfrac{2^2}{1^2+1}\cdot\dfrac{3^2}{3^2+1}\cdot\dfrac{5^2}{5^2+1}\cdots\cdots=\dfrac{\pi^2}{15}.$

6. (i) Prove that $\cos x+\tan z\sin x$

$$=\left(1+\frac{2x}{\pi-2z}\right)\left(1-\frac{2x}{\pi+2z}\right)\left(1+\frac{2x}{3\pi-2z}\right)\left(1-\frac{2x}{3\pi+2z}\right)\cdots\cdots$$

(ii) Hence deduce that

$$\tan z=\frac{2}{\pi-2z}-\frac{2}{\pi+2z}+\frac{2}{3\pi-2z}-\frac{2}{3\pi+2z}+\cdots\cdots$$

$$=8\sum_1^\infty\frac{z}{(2r-1)^2\pi^2-4z^2}.\quad z\neq(2r-1)\tfrac{1}{2}\pi$$

[*C. H. 1957.*]

7. (i) Prove that $\cos x-\cot z\sin x$

$$=\left(1-\frac{x}{z}\right)\left(1+\frac{x}{\pi-z}\right)\left(1-\frac{x}{\pi+z}\right)\left(1+\frac{x}{2\pi-z}\right)\left(1-\frac{x}{2\pi+z}\right)\cdots$$

(ii) Hence deduce that

$$\cot z=\frac{1}{z}-\frac{1}{\pi-z}+\frac{1}{\pi+z}-\frac{1}{2\pi-z}+\frac{1}{2\pi+z}$$

$$=\frac{1}{z}+2\sum_1^\infty\frac{z}{z^2-r^2\pi^2}.\quad z\neq r\pi$$

[*C. H. 1942, '57.*]

8. Show that

(i) $1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{5}+\dfrac{1}{7}-\dfrac{1}{8}+\dfrac{1}{10}-\cdots=\dfrac{\pi}{3\sqrt{3}}.$

[*Put $\frac{1}{6}\pi$ for z in the series for tan z, in Ex. 6(i).*]

(ii) $1-\dfrac{1}{5}+\dfrac{1}{7}-\dfrac{1}{11}+\dfrac{1}{13}-\dfrac{1}{17}+\dfrac{1}{19}-\cdots=\dfrac{\pi}{2\sqrt{3}}.$

[*C. H. 1935.*]

[*Put $\frac{1}{3}\pi$ for z in the series for cot z, in Ex. 7(ii).*]

(iii) $1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\cdots=\frac{\pi}{8}(\sqrt{2}+1)$.

[*Put* $\frac{1}{8}\pi$ *for* z *in the series for* $\cot z$, *in Ex.* 7(*ii*).]

9. Prove that

(i) $$\operatorname{cosec}\theta=\frac{1}{\theta}-\frac{1}{\theta-\pi}-\frac{1}{\theta+\pi}+\frac{1}{\theta-2\pi}+\frac{1}{\theta+2\pi}-\frac{1}{\theta-3\pi}-\frac{1}{\theta+3\pi}+\cdots$$
$$=\frac{1}{\theta}+2\theta\sum_{n=1}^{n=\infty}\frac{(-1)^n}{\theta^2-n^2\pi^2}.$$ [*C. H.* 1944, '48.]

[*Use* $\operatorname{cosec}\theta=\frac{1}{2}(\tan\frac{1}{2}\theta+\cot\frac{1}{2}\theta)$.]

(ii) $$\frac{1}{4\pi}\sec\theta=\frac{1}{\pi^2-4\theta^2}-\frac{3}{3^2\pi^2-4\theta^2}+\frac{5}{5^2\pi^2-4\theta^2}-\cdots$$
[*C. H.* 1936.]

[*Use* $2\sec\theta=\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)+\cot(\frac{1}{4}\pi+\frac{1}{2}\theta)$.]

10. Prove that

(i) $$\frac{1}{4}\sec^2\theta=\frac{1}{(\pi-2\theta)^2}+\frac{1}{(\pi+2\theta)^2}+\frac{1}{(3\pi-2\theta)^2}+\frac{1}{(3\pi+2\theta)^2}+\cdots$$

[*Put* $\theta+h$ *for* z *in the series for* $\tan z$ *in Ex.* 6(*ii*) *and equate the coefficients of* h.]

(ii) $$\operatorname{cosec}^2\theta=\frac{1}{\theta^2}+\frac{1}{(\theta-\pi)^2}+\frac{1}{(\theta+\pi)^2}+\frac{1}{(\theta-2\pi)^2}+\frac{1}{(\theta+2\pi)^2}+\cdots$$

[*Put* $\theta+h$ *for* z *in the series for* $\cot z$ *in Ex.* 7(*ii*) *and equate the coefficients of* h.]

11. Show that

(i) $$\sin x+\cos x=\left(1+\frac{4x}{\pi}\right)\left(1-\frac{4x}{3\pi}\right)\left(1+\frac{4x}{5\pi}\right)\left(1-\frac{4x}{7\pi}\right)\cdots$$

[*Put* $z=\frac{1}{4}\pi$ *in Ex.* 6(*i*).]

(ii) $\sin x + \cos x = \dfrac{\pi + 4x}{2\sqrt{2}}\left\{1 - \dfrac{(\pi + 4x)^2}{4^2\pi^2}\right\}\left\{1 - \dfrac{(\pi + 4x)^2}{8^2\pi^2}\right\}\cdots$

[*Put* $\sin x + \cos x = \sqrt{2}\sin(x + \frac{1}{4}\pi)$; *then factorise.*]

(iii) From the factors for $\sin x + \cos x$, deduce that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

[*Take logarithm of the first result and equate the coefficients of* x^2.]

12. Deduce the expression for $\sin x$ in factors from that for $\cos x$. [*C. H. 1933.*]

$$\left[\textit{Use } \frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cdots \right]$$

13. Prove that

$$\left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right)\cdots = \frac{1}{\pi}\sinh\pi.$$

14. Show that

$$\frac{\sin\theta}{\theta} = 1 - \frac{1}{\pi^2}\cdot\frac{\theta^2}{1^2} + \frac{1}{\pi^4}\cdot\frac{\theta^2(\theta^2 - \pi^2)}{1^2.2^2}$$

$$- \frac{1}{\pi^6}\cdot\frac{\theta^2(\theta^2 - \pi^2)(\theta^2 - 2^2\pi^2)}{1^2.2^2.3^2} + \cdots\cdots$$

and deduce that

$$\frac{1}{1^2}\cdot\frac{1}{2^2} + \left(\frac{1}{1^2} + \frac{1}{2^2}\right)\cdot\frac{1}{3^2} + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right)\cdot\frac{1}{4^2} + \cdots = \frac{\pi^4}{120}.$$

[*C. H. 1933.*]

15. Prove that

$$\left(1 + \frac{2}{1+1^2} + \frac{2}{1+2^2} + \frac{2}{1+3^2} + \cdots\right)\left(\frac{1}{4+1^2} + \frac{1}{4+3^2} + \frac{1}{4+5^2} + \cdots\right) = \frac{\pi^2}{8}.$$

[*C. H. 1931, '57.*]

16. Taking the infinite product forms as definitions of $\sin\theta$ and $\cos\theta$, prove that they are periodic.

17. Show that

$$\frac{1}{1^2+x^2}+\frac{1}{3^2+x^2}+\frac{1}{5^2+x^2}+\cdots=\frac{\pi}{4x}\tanh\frac{\pi}{2}x.$$

[*Put $z=\frac{1}{2}ix\pi$ in Ex. 6(ii).*]

18. Prove that

$$\frac{\left(\frac{\pi^2}{4}+1\right)\left(\frac{\pi^2}{4}+\frac{1}{9}\right)\left(\frac{\pi^2}{4}+\frac{1}{25}\right)\cdots\cdots}{\left(\frac{\pi^2}{4}+\frac{1}{4}\right)\left(\frac{\pi^2}{4}+\frac{1}{16}\right)\left(\frac{\pi^2}{4}+\frac{1}{36}\right)\cdots\cdots}=\frac{e^2+1}{e^2-1}.$$

[*Putting $x=1$ in the factors for sinh x and cosh x, left side reduces to $\frac{\cosh(1)}{\sinh(1)}$ i.e., to coth (1).*]

19. Show that

(i) $$\tan^{-1}(\tanh y\cot x)=\tan^{-1}\frac{y}{x}-\sum_{1}^{\infty}\tan^{-1}\frac{2xy}{n^2\pi^2-x^2+y^2}.$$

[*Take logarithm of the factors of sin $(x+iy)$ and equate the imaginary parts on both sides.*]

(ii) Hence deduce that

$$\sum_{1}^{\infty}\tan^{-1}\frac{1}{n^2\pi^2}=\frac{1}{4}\pi-\tan^{-1}\left(\tanh\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}\right).$$

20. Show that

$$\cos\left(\frac{\pi}{2}\sin\theta\right)=\frac{\pi}{4}\cos^2\theta\left(1+\frac{\cos^2\theta}{2.4}\right)\left(1+\frac{\cos^2\theta}{4.6}\right)\cdots$$

21. Show that

$$\cos\frac{\pi}{2}\theta=1-\theta^2-(1-\theta^2)\frac{\theta^2}{9}-(1-\theta^2)\left(1-\frac{\theta^2}{9}\right)\frac{\theta^2}{25}-\cdots$$

and hence deduce that

$$\frac{1}{3^2}+\left(1+\frac{1}{3^2}\right)\frac{1}{5^2}+\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{1}{7^2}+\cdots=\frac{\pi^4}{384}.$$

RESOLUTION INTO FACTORS (*Contd.*)

103. It is known from the Theory of Equations that if a be a root of the equation $f(x)=0$, then $x-a$ is a factor of the expression $f(x)$; and if $f(x)=0$ be an equation of the nth degree possessing the n roots $a_1,\ a_2,\ldots a_n$ and if the coefficient of the highest degree term in $f(x)$ be unity, then $f(x)=(x-a_1)(x-a_2)\cdots\cdots(x-a_n)$.

This principle will be made use of in the following articles.

104. Resolution of $x^{2n}-2x^n\cos n\theta+1$ into factors.

Let us first find the roots of the equation

$$x^{2n}-2x^n\cos n\theta+1=0.$$

The equation $x^{2n}-2x^n\cos n\theta+1=0$ may be written as

$$x^{2n}-2x^n\cos n\theta+\cos^2 n\theta+\sin^2 n\theta=0,$$

or, $(x^n-\cos n\theta)^2=-\sin^2 n\theta$,

therefore, $x^n-\cos n\theta=\pm i\sin n\theta$;

hence, $x=(\cos n\theta\pm i\sin n\theta)^{\frac{1}{n}}$.

As in Art. 49, the values of the right-hand side expression are the $2n$ quantities

$$\cos\frac{2r\pi+n\theta}{n}\pm i\sin\frac{2r\pi+n\theta}{n},\quad r=0,\ 1,\ 2,\ldots n-1.$$

Hence, the roots of the equation $x^{2n}-2x^n\cos n\theta+1=0$ are

$$\cos\theta\pm i\sin\theta,\ \cos\left\{\ +\frac{2\pi}{n}\right\}\pm i\sin\left\{\theta+\frac{2\pi}{n}\right\}\cdots\cdots$$

$$\cdots\cdots\cos\left\{\theta+\frac{2(n-1)\pi}{n}\right\}\pm i\sin\left\{\theta+\frac{2(n-1)\pi}{n}\right\}.$$

The factors corresponding to the first pair of roots are

$$(x-\cos\theta-i\sin\theta)(x-\cos\theta+i\sin\theta),$$

i.e., $\quad (x-\cos\theta)^2+\sin^2\theta,$

i.e., the quadratic factor, $x^2-2x\cos\theta+1$.

Similarly, the second, third,...... nth pairs of the above roots give as quadratic factors the expressions

$$x^2-2x\cos\left\{\theta+\frac{2\pi}{n}\right\}+1$$

$$x^2-2x\cos\left\{\theta+\frac{4\pi}{n}\right\}+1$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$x^2-2x\cos\left\{\theta+\frac{2(n-1)\pi}{n}\right\}+1.$$

Hence, $x^{2n}-2x^n\cos n\theta+1$

$$=\left\{x^2-2x\cos\theta+1\right\}\left\{x^2-2x\cos\left(\theta+\frac{2\pi}{n}\right)+1\right\}\cdots$$

$$\cdots\cdots\left\{x^2-2x\cos\left(\theta+\frac{2n-2}{n}\pi\right)+1\right\}. \qquad \cdots \quad (1)$$

The above relation may be shortly written as

$$\mathbf{x^{2n}-2x^n\cos n\theta+1=\prod_{r=0}^{n-1}\left\{x^2-2x\cos\left(\theta+\frac{2r\pi}{n}\right)+1\right\}}. \quad (2)$$

N. B. The symbol $\prod\limits_{r=0}^{n-1}$ stands for the product of all the factors corresponding to the integral values of r from 0 to $n-1$.

Replacing x by $\dfrac{x}{a}$ and multiplying by a^{2n}, it may be shown that

$$\mathbf{x^{2n}-2a^nx^n\cos n\theta+a^{2n}=\prod_{r=0}^{n-1}\left\{x^2-2ax\cos\left(\theta+\frac{2r\pi}{n}\right)+a^2\right\}} (3)$$

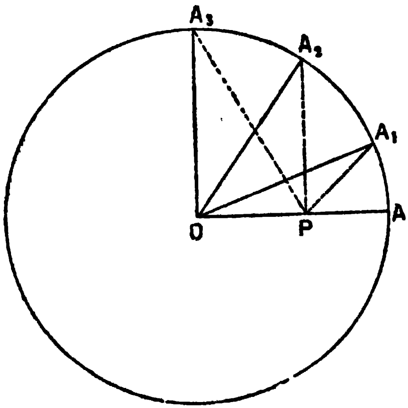
105. Geometrical Applications.

A geometrical meaning may be given to the equation (3) of the above article.

Let O be the centre of a circle of radius a, and P any point within or without it, such that $OP=x$; and let the whole circumference be divided into n equal arcs by the points $A_1, A_2, A_3, \ldots\ldots\ldots A_n$.

Then $\angle A_1OA_2 = \angle A_2OA_3 = \angle A_3OA_4 = \cdots = \dfrac{2\pi}{n}$.

Join O and P to the points of division $A_1, A_2, \ldots\ldots A_n$;



Let $\angle POA_1 = \theta$; then $\angle POA_2 = \theta + \dfrac{2\pi}{n}$;

$$\angle POA_3 = \theta + \frac{4\pi}{n}.$$

We have, $PA_1^2 = OP^2 + OA_1^2 - 2OP.OA_1 \cos POA_1$

$$= x^2 - 2ax \cos\theta + a^2.$$

$$PA_2^2 = OP^2 + OA_2^2 - 2OP.OA_2 \cos POA_2$$

$$= x^2 - 2ax \cos\left(\theta + \frac{2\pi}{n}\right) + a^2.$$

Similarly, $PA_3^2 = x^2 - 2ax \cos\left(\theta + \frac{4\pi}{n}\right) + a^2$,

$$PA_n^2 = x^2 - 2ax \cos\left(\theta + \frac{2n-2}{n}\pi\right) + a^2.$$

Hence, $PA_1^2.PA_2^2 \ldots\ldots PA_n^2$

$$= \left\{x^2 - 2ax \cos\theta + a^2\right\}\left\{x^2 - 2ax \cos\left(\theta + \frac{2\pi}{n}\right) + a^2\right\}\ldots\ldots$$

$$\ldots\left\{x^2 - 2ax \cos\left(\theta + \frac{2n-2}{n}\pi\right) + a^2\right\}$$

$$= x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$$

$$= OP^{2n} - 2OA_1^n OP^n \cos n\theta + OA_1^{2n}. \qquad \cdots \quad (4)$$

This is known as De Moivre's property of the circle.

Particular cases :

(i) When P is on the circumference of the circle,

$$OP = a = OA_1.$$

Therefore, $OP^{2n} - 2OA_1^n.OP^n \cos n\theta + OA_1^{2n}$

$$= 2OA_1^{2n}(1 - \cos n\theta) = 4OA_1^{2n} \sin^2 \frac{n\theta}{2}.$$

Hence, the above relation becomes

$$PA_1.PA_2\ldots\ldots PA_n = 2OA_1^n \sin \frac{n\theta}{2}.$$

(ii) When P lies on OA_1; then $\theta = 0$.

Therefore, $PA_1^2.PA_2^2\ldots\ldots PA_n^2 = x^{2n} - 2a^n x^n + a^{2n}$

$$= (x^n - a^n)^2.$$

$$\therefore \quad PA_1.PA_2\ldots\ldots PA_n = x^n - a^n, \text{ or, } a^n - x^n.$$

The first of the two values should be taken when x is $> a$, *i.e.*, when OP is $> OA$; in other words, when P lies outside the circle.

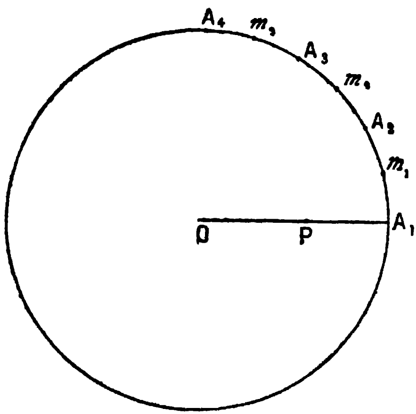
The second value should be taken when x is $< a$, *i.e.*, when OP is $< OA$; in other words, when P lies inside the circle.

Therefore, $PA_1.PA_2 \ldots\ldots PA_n = x^n \sim a^n$. ... (5)

Again, let $m_1, m_2, m_3, \ldots\ldots$ be the middle points of the arcs $A_1A_2, A_2A_3, A_3A_4, \ldots\ldots$ so that the whole circumference is now divided into $2n$ equal arcs by the points $A_1, m_1, A_2, m_2, \ldots\ldots$

Hence, by (5),

$$PA_1.Pm_1.PA_2.Pm_2 \ldots \text{ to } 2n \text{ factors} = x^{2n} \sim a^{2n} \quad \ldots \quad (6)$$



Dividing (6) by (5),

$$Pm_1.Pm_2 \ldots\ldots Pm_n = x^n + a^n. \quad \cdots \quad (7)$$

This is known as Cote's property of the circle.

Obs. Since $\angle POM_1 = \frac{2\pi}{2n} = \frac{\pi}{n}$, hence the relation (7) can be deduced directly from equation (4) by putting $\theta = \frac{\pi}{n}$ and $PA_1, PA_2, \ldots\ldots = Pm_1, Pm_2 \ldots\ldots$

106. Resolution of $x^n - 1$ into factors.

Let us first find the roots of the equation $x^n - 1 = 0$.

$$x^n - 1 = 0, \text{ or, } x^n = 1 = \cos 0 + i \sin 0.$$

$$\therefore \quad x = (\cos 0 + i \sin 0)^{\frac{1}{n}}.$$

As in Art. 49, the values of the right-hand side expression are the n quantities,

$$\cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}, \; r = 0, 1, 2, \ldots\ldots n-1.$$

Hence, the roots of the equation $x^n - 1 = 0$ are

$$\cos 0 + i \sin 0, \; \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, \; \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n}, \ldots$$

$$\cos \frac{2n\pi - 4\pi}{n} + i \sin \frac{2n\pi - 4\pi}{n}, \; \cos \frac{2n\pi - 2\pi}{n} + i \sin \frac{2n\pi - 2\pi}{n}.$$

The last term in the above series $= \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n}$.

The last but one term $= \cos \frac{4\pi}{n} - i \sin \frac{4\pi}{n}$, and so on.

Thus, we get conjugate complex roots for $r = 1$ and $n-1$, $r = 2$ and $n-2$, and so on.

First, let n be even.

Now, for $r = \frac{n}{2}$, the value of the expression is

$$\cos \pi + i \sin \pi = -1.$$

Thus, when n is *even*, $x^n - 1 = 0$ has two real roots and $\left(\frac{n}{2} - 1\right)$ pairs of conjugate imaginary roots,* and these are

$$\pm 1, \; \cos \frac{2\pi}{n} \pm i \sin \frac{2\pi}{n}, \; \cos \frac{4\pi}{n} \pm i \sin \frac{4\pi}{n}, \ldots\ldots$$

$$\cos \frac{n-2}{n}\pi \pm i \sin \frac{n-2}{n}\pi.$$

*It is a well-known theorem in the Theory of Equations that imaginary roots occur in conjugate pairs.

The factors corresponding to the real roots are $(x-1)$ and $(x+1)$, *i.e.*, the quadratic factor x^2-1.

Those corresponding to the first pair of complex roots are

$$\left(x-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}\right) \text{ and } \left(x-\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right),$$

i.e., the quadratic factor $x^2-2x\cos\frac{2\pi}{n}+1$, and so on.

Hence, *when n is even,*

$$\mathbf{x^n-1}$$

$$=\left(x^2-1\right)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\cdots\cdots\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right)$$

$$=\left(\mathbf{x^2-1}\right)\prod_{r=1}^{\frac{n}{2}-1}\left(\mathbf{x^2-2x\cos\frac{2r\pi}{n}+1}\right). \qquad \cdots \quad (8)$$

Secondly, let n be ***odd.***

When n is odd, $x^n-1=0$ has one real root and $\frac{n-1}{2}$ pairs of conjugate imaginary roots and these are

$$1,\ \cos\frac{2\pi}{n}\pm i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}\pm i\sin\frac{4\pi}{n},\cdots\cdots$$

$$\cos\frac{n-1}{n}\pi\pm i\sin\frac{n-1}{n}\pi.$$

Combining, as before, factors corresponding to the imaginary roots, we get *when n is odd,*

$$\mathbf{x^n-1}=\left(x-1\right)\left\{x^2-2x\cos\frac{2\pi}{n}+1\right\}\cdots\cdots$$

$$\cdots\left\{x^2-2x\cos\frac{n-1}{n}\pi+1\right\}$$

$$=\left(\mathbf{x-1}\right)\prod_{r=1}^{\frac{n-1}{2}}\left(\mathbf{x^2-2x\cos\frac{2r\pi}{n}+1}\right). \qquad \cdots \quad (9)$$

Cor. Expression of sin θ into factors. *(Alternative method)*

From (8), writing $2m$ for n (n being an even integer), and dividing both sides by x^m,

$$x^m - x^{-m} = (x - x^{-1}) \prod_{r=1}^{m-1} \left(x + x^{-1} - 2\cos\frac{r\pi}{m}\right).$$

Now, putting $x = \cos\frac{\theta}{m} + i\sin\frac{\theta}{m}$, so that $x + x^{-1} = 2\cos\frac{\theta}{m}$,

$x - x^{-1} = 2i\sin\frac{\theta}{m}$, $x^m - x^{-m} = 2i\sin\theta$, and noticing that

$\cos\frac{\theta}{m} - \cos\frac{r\pi}{m} = 2\sin^2\frac{r\pi}{2m} - 2\sin^2\frac{\theta}{2m}$,

the above reduces to $\left(\text{after dividing both sides by }\sin\frac{\theta}{m}\right)$,

$$\frac{\sin\theta}{\sin\frac{\theta}{m}} = 2^{2m-2}\left\{\sin^2\frac{\pi}{2m} - \sin^2\frac{\theta}{2m}\right\}\left\{\sin^2\frac{2\pi}{2m} - \sin^2\frac{\theta}{2m}\right\}\cdots\cdots$$

$$\cdots\cdots\left\{\sin^2\frac{(m-1)\pi}{2m} - \sin^2\frac{\theta}{2m}\right\}. \qquad \text{(i)}$$

Making $\theta \to 0$,

$\because \underset{\theta\to 0}{Lt}\ \dfrac{\sin\theta}{\sin\frac{\theta}{m}} = m$, we get,

$$m = 2^{2m-2}\sin^2\frac{\pi}{2m}\sin^2\frac{2\pi}{2m}\cdots\sin^2\frac{(m-1)\pi}{2m}. \qquad \text{(ii)}$$

Dividing (i) by (ii),

$$\frac{\sin\theta}{m\sin\frac{\theta}{m}} = \left\{1 - \frac{\sin^2\frac{\theta}{2m}}{\sin^2\frac{\pi}{2m}}\right\}\left\{1 - \frac{\sin^2\frac{\theta}{2m}}{\sin^2\frac{2\pi}{2m}}\right\}\cdots\cdots\cdots \text{ to } m-1 \text{ factors.}$$

As this is true for all integral values of m, make m an infinitely large positive integer.

Thus, since, $\underset{m\to\infty}{Lt}\ m\sin\frac{\theta}{m} = \theta$,

and $\qquad \underset{m\to\infty}{Lt}\ \dfrac{\sin\frac{\theta}{2m}}{\sin\frac{r\pi}{2m}} = \dfrac{\theta}{r\pi}$,

we ultimately get

$$\frac{\sin\theta}{\theta} = \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\cdots\cdots$$

the number of factors on the right-hand side being infinite.

107. Resolution of x^n+1 into factors.

Let us first find the roots of the equation $x^n+1=0$.

$$x^n+1=0, \text{ or, } x^n=-1=\cos\pi+i\sin\pi.$$

Therefore, $x=(\cos\pi+i\sin\pi)^{\frac{1}{n}}$.

By Art. 49, the values of the right-hand side expression are

$$\cos\frac{2r\pi+\pi}{n}+i\sin\frac{2r\pi+\pi}{n},\ r=0,1,2,\ldots n-1. \quad \cdots \text{(A)}$$

Hence, the roots of the equation $x^n+1=0$ are

$$\cos\frac{\pi}{n}+i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n},\ldots\ldots$$

$$\cos\frac{2n\pi-3\pi}{n}+i\sin\frac{2n\pi-3\pi}{n},$$

$$\cos\frac{2n\pi-\pi}{n}+i\sin\frac{2n\pi-\pi}{n}.$$

The last term in the above series

$$=\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}.$$

The last but one term

$$=\cos\frac{3\pi}{n}-i\sin\frac{3\pi}{n}, \text{ and so on.}$$

Thus, we get conjugate imaginary roots for $r=0$ and $n-1$, $r=1$ and $n-2$, and so on.

First, let n be even.

When n is even, $x^n+1=0$ has $\frac{n}{2}$ pairs of conjugate imaginary roots and these are

$$\cos\frac{\pi}{n}\pm i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n}\pm i\sin\frac{3\pi}{n}.$$

$$\cdot\cos\frac{(n-1)\pi}{n}\pm i\sin\frac{(n-1)\pi}{n}.$$

The factors corresponding to the first pair of imaginary roots are

$$\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right) \text{ and } \left(x-\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)$$

i.e., the quadratic factor $x^2-2x\cos\frac{\pi}{n}+1$.

Similarly, the factors corresponding to the other pairs of imaginary roots are obtained.

Thus, *when n is even,*

$$\mathbf{x^n+1}=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)$$

$$\cdots\cdots\left(x^2-2x\cos\frac{(n-1)\pi}{n}+1\right),$$

i.e., $$=\prod_{r=0}^{\frac{n}{2}-1}\left(\mathbf{x^2-2x\cos\frac{2r+1}{n}\pi+1}\right). \quad\cdots\quad (10)$$

Secondly, let n be odd.

Now, for $r=\frac{n-1}{2}$, the value of the expression (A) is $\cos\pi+i\sin\pi=-1$.

Thus, when n is *odd*, $x^n+1=0$ has *one* real root and $\frac{n-1}{2}$ pair of imaginary roots and these are

$$=-1,\ \cos\frac{\pi}{n}\pm i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n}\pm i\sin\frac{3\pi}{n},\cdots\cdots$$

$$\cdots\cdots\cos\frac{(n-2)\pi}{n}\pm i\sin\frac{(n-2)\pi}{n}.$$

Combining, as before, the factors corresponding to the pair of imaginary roots, we get *when n is odd,*

$$\mathbf{x^n+1}=\left(x+1\right)\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\cdots$$

$$\cdots\cdots\left(x^2-2x\cos\frac{(n-2)\pi}{n}+1\right),$$

i.e., $$=\mathbf{(x+1)}\prod_{r=0}^{\frac{n-3}{2}}\left(\mathbf{x^2-2x\cos\frac{2r+1}{n}\pi+1}\right). \quad\cdots\quad (11)$$

Cor. Expression of cos θ into factors. *(Alternative method)*

From (10), replacing n by $2m$, putting $x=\cos\frac{\theta}{m}+i\sin\frac{\theta}{m}$, and proceeding exactly as in Cor., Art. 106, we get

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots\cdots \text{ ad. inf.}$$

108. Ex. 1. *Express as a product of n factors*

$\cos n\phi-\cos n\theta$.

Dividing both sides of equation (2), Art. 104, by x^n, we have

$$x^n-2\cos n\theta+\frac{1}{x^n}=\prod_{r=0}^{n-1}\left\{x-2\cos\left(\theta+\frac{2r\pi}{n}\right)+\frac{1}{x}\right\}.$$

Puttting in this equation $x=e^{i\Phi}$, so that $x^{-1}=e^{-i\Phi}$

and hence $\quad x+x^{-1}=e^{i\Phi}+e^{-i\Phi}=2\cos\phi,$

$$x^n+x^{-n}=e^{in\Phi}+e^{-in\Phi}=2\cos n\phi,$$

we have,

$$=\prod_{r=0}^{n-1}\left\{2\cos\phi-2\cos\left(\theta+\frac{2r\pi}{n}\right)\right\}$$

$$=2^n\prod_{r=0}^{n-1}\left\{\cos\phi-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\}.$$

$$\therefore\quad \cos n\phi-\cos n\theta=2^{n-1}\prod_{r=0}^{n-1}\left\{\cos\phi-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\}.$$

Ex. 2. *Show that*

$$\text{(i)}\ \sin n\phi=2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\cdots\sin\left(\phi+\frac{n-1}{n}\pi\right).$$

[*C. H. 1935.*]

$$\text{(ii)}\ \cos n\phi=2^{n-1}\sin\left(\phi+\frac{\pi}{2n}\right)\sin\left(\phi+\frac{3\pi}{2n}\right)\cdots\sin\left(\phi+\frac{2n-1}{2n}\pi\right).$$

(i) Putting $x=1$ and $\theta=2\phi$ in the equation (1) of Art. 104, we have $2-2\cos 2n\phi=(2-2\cos 2\phi)\left\{2-2\cos\left(2\phi+\frac{2\pi}{n}\right)\right\}\cdots\cdots$ to n factors

i.e., $\quad 4\sin^2 n\phi=4\sin^2\phi.4\sin^2\left(\phi+\frac{\pi}{n}\right)\cdots$ to n factors.

Now, extracting the square root of both sides, the required result follows.

(ii) This result follows from (i) by putting $\phi+\frac{\pi}{2n}$ for ϕ.

Ex. 3. *Prove that when n is odd*

$$\tan n\phi=(-1)^{\frac{n-1}{2}}\tan\phi\tan\left(\phi+\frac{\pi}{n}\right)\cdots\tan\left(\phi+\frac{n-1}{n}\pi\right).$$

By changing ϕ into $\phi+\frac{\pi}{2}$ in the result of Ex. 2(i) above, we have

$$\sin\left(n\phi+\frac{n\pi}{2}\right)=2^{n-1}\sin\left[\phi+\frac{\pi}{2}\right]\sin\left[\phi+\frac{\pi}{2}+\frac{\pi}{n}\right]\cdots \text{ to } n \text{ factors.}$$

$$\text{Left side}=\sin n\phi\cos\frac{n\pi}{2}+\cos n\phi\sin\frac{n\pi}{2}$$

$$=(-1)^{\frac{n-1}{2}}\cos n\phi, \text{ since } n \text{ is odd.}$$

$$\text{Right side}=2^{n-1}\cos\phi\cos\left[\phi+\frac{\pi}{n}\right]\cdots\cdots \text{ to } n \text{ factors.}$$

$$\therefore\quad (-1)^{\frac{n-1}{2}}\cos n\phi=2^{n-1}\cos\phi\cos\left[\phi+\frac{\pi}{n}\right]\cdots\cdots\cos\left[\phi+\frac{n-1}{n}\pi\right].$$

The required result follows by dividing the result of Ex. 2(i) by this.

Ex. 4. *Prove that*

$$2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\cdots=\sqrt{n}. \qquad [\textit{C. H. 1939, '41, '43.}]$$

From the result of Ex. 2 (i), we have,

$$\frac{\sin n\phi}{\sin\phi}=2^{n-1}\sin\left[\phi+\frac{\pi}{n}\right]\sin\left[\phi+\frac{2\pi}{n}\right]\cdots\cdots\sin\left[\phi+\frac{n-1}{n}\pi\right].$$

Making $\phi\to0$ and noting that as ϕ tends to 0, $\frac{\sin n\phi}{\sin\phi}$ tends to n,

$$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\cdots\sin\frac{n-1}{n}\pi.$$

Now, in the right-hand side, factors equidistant from the beginning and end are equal.

$$\therefore\ 2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\cdots=\sqrt{n},$$

the last factor being $\sin\frac{n-1}{2n}\pi$ when n is *odd*, and $\sin\frac{n-2}{2n}\pi$ when n is *even*.

EXAMPLES XIV(b)

1. Prove that

(i) $2^{n-1} \sin \dfrac{\pi}{2n} \sin \dfrac{3\pi}{2n} \cdots\cdots \sin \dfrac{(2n-1)\pi}{2n} = 1.$

[*Put* $\phi = \dfrac{\pi}{2n}$ *in Ex. 2(i) worked out.*]

(ii) $2^{n-1} \sin \dfrac{\pi}{n} \sin \dfrac{2\pi}{n} \cdots \sin \dfrac{(n-1)\pi}{n} = n.$ [*C. H. 1946.*]

(iii) $2^{n-1} \cos \dfrac{\pi}{n} \cos \dfrac{2\pi}{n} \cdots\cdots \cos \dfrac{(n-1)\pi}{n} = 0,\ 1$ or -1

according as n is even or of the form $4p+1$ or $4p-1$.

[*C. H. 1940.*]

(iv) $\tan \phi \tan \left[\phi + \dfrac{\pi}{n}\right] \tan \left[\phi + \dfrac{2\pi}{n}\right] \cdots\cdots$ to n factors

$= (-1)^{\frac{1}{2}n}$, if n be *even.* [*C. H. 1938.*]

2. Prove that

$$\prod_{r=0}^{n-1} \left\{\cos \theta - \cos \frac{2r\pi}{n}\right\} + \prod_{r=0}^{n-1} \left\{1 - \cos \left(\theta + \frac{2r\pi}{n}\right)\right\} = 0.$$

3. Prove that

$$\frac{x^n - a^n \cos n\theta}{x^{2n} - 2a^n x^n \cos n\theta + a^{2n}} = \frac{1}{nx^{n-1}} \sum_{r=0}^{n-1} \frac{x - a \cos \left(\theta + \dfrac{2r\pi}{n}\right)}{x^2 - 2ax \cos \left(\theta + \dfrac{2r\pi}{n}\right) + a^2}.$$

4. Prove that

$$\cosh n\phi - \cos n\theta = 2^{n-1} \prod_{r=0}^{n-1} \left\{\cosh \phi - \cos \left(\theta + \frac{2r\pi}{n}\right)\right\}.$$

[*C. H. 1957.*]

5. $A_1, A_2, \ldots A_{2n}$ are the vertices of a regular polygon of $2n$ sides inscribed in a circle of radius a.

(i) If O be the mid-point of the arc $A_1 A_{2n}$, prove that

$$OA_1 . OA_2 \ldots\ldots OA_n = \sqrt{2}a^n.$$

(ii) Show that $A_1A_2 . A_1A_3 \ldots\ldots A_1A_n = a^{n-1}\sqrt{n}.$

6. $A_1, A_2, \ldots A_{2n+1}$ are the vertices of a regular polygon of $(2n+1)$ sides inscribed in a circle of radius a. If the diameter through A_{n+1} meets the circle again in Q, show that

$$QA_1 \cdot QA_2 \ldots\ldots QA_n = a^n.$$

7. If $\rho_1, \rho_2, \ldots \rho_n$ be the distances of a point P in the plane of a regular polygon of n sides from the vertices, prove that

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \cdots + \frac{1}{\rho_n^2} = \frac{r^{2n} - a^{2n}}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}} \cdot \frac{n}{r^2 - a^2},$$

a being the radius of the circum-circle, r the distance of P from the centre O and θ the angle between OP and ρ_1.

[*C. H. 1931.*]

8. The product of all the straight lines that can be drawn from one of the angles of a regular polygon of n sides inscribed in a circle whose radius is a, to all the other angular points is na^{n-1}.

9. Prove that

$$\frac{\sin 2a \sin 4a \ldots\ldots \sin (2n-2)a}{\sin a \sin 3a \ldots\ldots \sin (2n-1)a} = n,$$

where $2na = \pi$.

[*Apply the result of Ex. 1 (i) & (ii).*]

10. Show that $\cot a \cot 3a \cot 5a \ldots\ldots \cot (2n+1)a = 1$, where $4(n+1)a = \pi$.

11. Prove that

$$(x+1)^{2n} + (x-1)^{2n} = 2 \prod_{r=1}^{n} \left[x^2 + \tan^2 \frac{(2r-1)\pi}{4n} \right].$$

12. Show that

$$\tan^{-1} (\cot nx \tanh ny)$$
$$= \tan^{-1} (\cot x \tanh y) + \tan^{-1} \left\{ \cot \left(x + \frac{\pi}{n} \right) \tanh y \right\}$$
$$+ \tan^{-1} \left\{ \cot \left(x + \frac{2\pi}{n} \right) \tanh y \right\} + \cdots \text{ to } n \text{ terms.}$$

[*C. H. 1935.*]

[*See Ex. 6, Art. 89 and Ex. 2(i), Art. 108.*]

CHAPTER XV

MISCELLANEOUS THEOREMS AND EXAMPLES

Sec. A.—IDENTITIES AND TRANSFORMATIONS

109. The following examples will illustrate some important types of trigonometrical identities and transformations.

Ex. 1. *Prove that* $\Sigma \cos(\alpha+\theta)\sin(\beta-\gamma)=0$.

$$\begin{aligned}\Sigma \cos(\alpha+\theta)\sin(\beta-\gamma) &= \Sigma(\cos\alpha\cos\theta-\sin\alpha\sin\theta)\sin(\beta-\gamma)\\ &=\cos\theta\,\Sigma\cos\alpha\sin(\beta-\gamma)\\ &\qquad-\sin\theta\,\Sigma\sin\alpha\sin(\beta-\gamma)\\ &=0,\end{aligned}$$

since, $\quad \Sigma\cos\alpha\sin(\beta-\gamma)=0$ and $\Sigma\sin\alpha\sin(\beta-\gamma)=0$.

Ex. 2. *If* $\cos^2x+\cos^2y+\cos^2z+2\cos x\cos y\cos z-1=0$,

show that $x\pm y\pm z=(2n+1)\pi$.

$$\begin{aligned}&\cos^2x+\cos^2y+\cos^2z+2\cos x\cos y\cos z-1\\ &\quad=\cos^2x+(\cos^2y+\cos^2z-1)+2\cos x\cos y\cos z\\ &\quad=\cos^2x+(\cos^2y-\sin^2z)+2\cos x\cos y\cos z\\ &\quad=\cos^2x+\cos(y+z)\cos(y-z)+\cos x\{\cos(y+z)+\cos(y-z)\}\\ &\quad=\{\cos x+\cos(y+z)\}\{\cos x+\cos(y-z)\}\\ &\quad=4\cos\frac{x+y+z}{2}\cos\frac{x-y-z}{2}\cos\frac{x+y-z}{2}\cos\frac{x-y+z}{2}.\end{aligned}$$

Since, the left side is equal to zero by the given condition,

$\therefore$ one of the quantities $\cos\dfrac{x\pm y\pm z}{2}=0$,

i.e., one of the four angles $\dfrac{x\pm y\pm z}{2}=(2n+1)\dfrac{\pi}{2}$,

i.e., $x\pm y\pm z=(2n+1)\pi$.

Ex. 3. *Prove that*

$$\Sigma \sin 2\alpha \sin (\beta-\gamma) = -\Sigma \sin (\beta+\gamma)\, \Sigma \sin (\beta-\gamma).$$

Right side $= -\{\sin\beta\cos\gamma+\cos\beta\sin\gamma+\sin\gamma\cos\alpha+\cos\gamma\sin\alpha$
$+\sin\alpha\cos\beta+\cos\alpha\sin\beta\}$
$\times\{\sin\beta\cos\gamma-\cos\beta\sin\gamma+\sin\gamma\cos\alpha-\cos\gamma\sin\alpha$
$+\sin\alpha\cos\beta-\cos\alpha\sin\beta\}$
$= -[\{\sin\beta\cos\gamma+\sin\gamma\cos\alpha+\sin\alpha\cos\beta\}^2$
$-\{\cos\beta\sin\gamma+\cos\gamma\sin\alpha+\cos\alpha\sin\beta\}^2]$.

Now, $\sin^2\beta\cos^2\gamma-\cos^2\beta\sin^2\gamma$

$$=\sin^2\beta\,(1-\sin^2\gamma)-(1-\sin^2\beta)\sin^2\gamma=\sin^2\beta-\sin^2\gamma\,;$$

hence, the algebraic sum of the square terms is zero; the product terms are equal to

$$\begin{aligned}&2\sin\alpha\cos\alpha\,(\sin\beta\cos\gamma-\cos\beta\sin\gamma)\\&\quad+2\sin\beta\cos\beta\,(\sin\gamma\cos\alpha-\cos\gamma\sin\alpha)\\&\quad+2\sin\gamma\cos\gamma\,(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\&=\Sigma\sin 2\alpha\sin(\beta-\gamma).\end{aligned}$$

Note. In this example, putting $\frac{1}{4}\pi+\alpha$, $\frac{1}{4}\pi+\beta$, $\frac{1}{4}\pi+\gamma$ for α, β, γ respectively, we obtain the identity

$$\Sigma \cos 2\alpha \sin (\beta-\gamma) = -\Sigma \cos (\beta+\gamma)\, \Sigma \sin (\beta-\gamma).$$ [*C. H. 1957.*]

Ex. 4. *Prove that if*

$$\frac{1-\tan B\tan C}{\cos^2 A}+\frac{1-\tan C\tan A}{\cos^2 B}=2\frac{1-\tan A\tan B}{\cos^2 C},$$

either $\tan A$, $\tan C$, $\tan B$ are in A.P., or, $A+B+C$ is an integral multiple of π.

Taking the denominators to the numerators and expressing them in terms of the tangent, the given relation becomes

$$(1+\tan^2 A)(1-\tan B\tan C)+(1+\tan^2 B)(1-\tan C\tan A)$$
$$=2\,(1+\tan^2 C)(1-\tan A\tan B).$$

This, when simplified, reduces to

$$(\tan A+\tan B+\tan C)(\tan A+\tan B-2\tan C)$$
$$-\tan A\tan B\tan C(\tan A+\tan B-2\tan C)=0.$$

$\therefore$ $(\tan A+\tan B-2\tan C)(\tan A+\tan B+\tan C$
$$-\tan A\tan B\tan C)=0\,;$$

$\therefore$ either $\tan A+\tan B-2\tan C=0$,

i.e., $\tan A$, $\tan C$, $\tan B$ are in A. P.,

or, $\tan A+\tan B+\tan C-\tan A\tan B\tan C=0$,

whence $\tan(A+B+C)=0$,

i.e., $A+B+C=n\pi$ (n being zero or any integer).

Ex. 5. *If $\cos A=\cos\theta\sin\phi$, $\cos B=\cos\phi\sin\psi$, $\cos C=\cos\psi\sin\theta$ and $A+B+C=\pi$, prove that $\tan\theta\tan\phi\tan\psi=1$.* [*C. H. 1935.*]

If $A+B+C=\pi$, then

$$\cos^2A+\cos^2B+\cos^2C=1-2\cos A\cos B\cos C. \qquad \cdots \quad (1)$$

Now, $\cos^2A+\cos^2B+\cos^2C$

$$=\cos^2\theta\sin^2\phi+\cos^2\phi\sin^2\psi+\cos^2\psi\sin^2\theta$$
$$=\sin^2\phi\,(1-\sin^2\theta)+\cos^2\phi\,(1-\cos^2\psi)+\cos^2\psi\sin^2\theta$$
$$=(\sin^2\phi+\cos^2\phi)-\sin^2\phi\sin^2\theta-\cos^2\phi\cos^2\psi+\cos^2\psi\sin^2\theta$$
$$=1-\sin^2\phi\sin^2\theta\,(\cos^2\psi+\sin^2\psi)-\cos^2\phi\cos^2\psi\,(\sin^2\theta+\cos^2\theta)$$
$$+\cos^2\psi\sin^2\theta\,(\sin^2\phi+\cos^2\phi)$$
$$=1-(\sin^2\theta\sin^2\phi\sin^2\psi+\cos^2\theta\cos^2\phi\cos^2\psi)$$
$$=1-2\cos\theta\sin\phi.\cos\phi\sin\psi\cos\psi\sin\theta, \text{ from (1).}$$

$\therefore$ $\sin^2\theta\sin^2\phi\sin^2\psi+\cos^2\theta\cos^2\phi\cos^2\psi$
$$-2\sin\theta\sin\phi\sin\psi\cos\theta\cos\phi\cos\psi=0.$$

$\therefore$ $(\sin\theta\sin\phi\sin\psi-\cos\theta\cos\phi\cos\psi)^2=0.$

$\therefore$ $\sin\theta\sin\phi\sin\psi=\cos\theta\cos\phi\cos\psi.$

$\therefore$ $\tan\theta\tan\phi\tan\psi=1.$

EXAMPLES XV(a)

1. (i) If $A+B+C=\pi$ and $\cos A=\cos B\cos C$, prove that $\cot B\cot C=\frac{1}{2}$. [*C. P. 1933.*]

(ii) Prove that the expression

$\sin^2(\theta+\alpha)+\sin^2(\theta+\beta)-2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$ is independent of θ.

(iii) If $\tan\frac{\theta}{2}=\tan^3\frac{\phi}{2}$, and $\tan\phi=2\tan\alpha$,

show that $\theta+\phi=2\alpha$.

2. If $\frac{\sin x}{a_1}=\frac{\sin 3x}{a_3}=\frac{\sin 5x}{a_5}$, show that

$$\frac{a_1-2a_3+a_5}{a_3}=\frac{a_3-3a_1}{a_1}.$$

3. If $A+B+C=\pi$, prove that

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}=0.$$

[*C. H. 1931.*]

4. Prove that if,

$$\begin{vmatrix} 1 & -\cos z & -\cos y \\ -\cos z & 1 & -\cos x \\ -\cos y & -\cos x & 1 \end{vmatrix}=0.$$

then $x\pm y\pm z=(2n+1)\pi$.

[*The determinant when expanded will reduce to Ex. 2 worked out.*]

5. If $\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$, prove that

$$\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3}$$

6. Show that

$$\Sigma \cot (2\alpha+\beta-3\gamma) \cot (2\beta+\gamma-3\alpha)=1.$$

7. If $\cos^2 x+\cos^2 y+\cos^2 z-2 \cos x \cos y \cos z=1$, then show that

$$x \pm y \pm z=2n\pi.$$

[*Proceed as in Ex. 2 worked out.*]

8. If $\alpha+\beta+\gamma=0$, prove that

$$\Sigma \sin 2\alpha=2\,(\Sigma \sin \alpha)(1+\Sigma \cos \alpha).$$

9. If $x+y+z=xyz$, then

$$\Sigma\left(\frac{3x-x^3}{1-3x^2}\right)=\Pi\left(\frac{3x-x^3}{1-3x^2}\right).$$

10. Show that

$$\begin{vmatrix} 1 & \cos\theta & \cos 2\theta & \cos 3\theta \\ \cos\theta & \cos 2\theta & \cos 3\theta & \cos 4\theta \\ \cos 2\theta & \cos 3\theta & \cos 4\theta & \cos 5\theta \\ \cos 3\theta & \cos 4\theta & \cos 5\theta & \cos 6\theta \end{vmatrix}=0.$$

[*Multiply the 2nd row by 2 cos θ, now if 3rd row be subtracted from the second, two rows would be identical.*]

11. Show that

$$\begin{vmatrix} \sin(\alpha+x) & \sin(\alpha+y) & \sin(\alpha+z) \\ \sin(\beta+x) & \sin(\beta+y) & \sin(\beta+z) \\ \sin(\gamma+x) & \sin(\gamma+y) & \sin(\gamma+z) \end{vmatrix}=0.$$

[*The given determinant is the product of two determinants* ($\sin\alpha$, $\cos\beta$, 0) *and* ($\cos x$, $\sin y$, 0).]

12. If $\alpha+\beta+\gamma=\pi$, and

$$\tan \tfrac{1}{2}(\beta+\gamma-\alpha) \tan \tfrac{1}{2}(\gamma+\alpha-\beta) \tan \tfrac{1}{2}(\alpha+\beta-\gamma)=1,$$

then $\quad 1+\cos\alpha+\cos\beta+\cos\gamma=0.$ [*C. H. 1934.*]

13. Granting that A, B, C are the three angles of a *real* plane triangle, prove that the two relations

$$\sin^2 A + \sin^2 B + \sin^2 C = \sin B \sin C + \sin C \sin A + \sin A \sin B$$

and $$\cot A + \cot B + \cot C = \sqrt{3}$$

are equivalent to each other and that either of them may be taken to characterise an *equilateral triangle.*

[*C. H. 1930 ; C. P. 1940.*]

14. If $A + B + C = \pi$, and if

$\cos A = \cot y \cot z$, $\cos B = \cot z \cot x$, $\cos C = \cot x \cot y$,

then show that $\cos^2 x + \cos^2 y + \cos^2 z = 1$.

15. Show that $\Sigma \sin^3 x \sin^3 (y - z)$

$$= 3 \sin x \sin y \sin z \sin (y - z) \sin (z - x) \sin (x - y).$$

[*$\Sigma \sin x \sin (y - z) = 0$; now apply the result "if $a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$."*]

16. If $(\sin x + \sin y + \sin z)^2 + (\cos x + \cos y + \cos z)^2 = 1$, show that two of the angles differ by $(2n + 1)\pi$.

17. If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$, prove that $\sin A = \sin B = \sin C = 2 \sin 18°$. [*C. P. 1939.*]

18. If $\Sigma (b + c) \tan \alpha = 0$ and $\Sigma a \tan \beta \tan \gamma = \Sigma a$, then prove that $\Sigma a \sin 2\alpha = 0$.

[*Obtain the proportional values of a, b, c from the first two equations by the Rule of Cross-Multiplication.*]

19. Prove that

$$\sin (\alpha + \beta) \sin (\alpha - \beta) \sin (\gamma + \delta) \sin (\gamma - \delta)$$
$$+ \sin (\beta + \gamma) \sin (\beta - \gamma) \sin (\alpha + \delta) \sin (\alpha - \delta)$$
$$+ \sin (\gamma + \alpha) \sin (\gamma - \alpha) \sin (\beta + \delta) \sin (\beta - \delta) = 0.$$

[*Put $a = \sin \alpha$, $b = \sin \beta$, $c = \sin \gamma$, $d = \sin \delta$; left side becomes $(a^2 - b^2)(c^2 - d^2) + (b^2 - c^2)(a^2 - d^2) + (c^2 - a^2)(b^2 - d^2)$ which is zero.*]

20. If $\tan \frac{1}{2}(y + z - x) \tan \frac{1}{2}(z + x - y) \tan \frac{1}{2}(x + y - z) = 1$, show that $\Sigma \sin 2x = 4 \cos x \cos y \cos z$.

21. If $\theta+\phi+\psi=\pi$, prove that

$$\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3}+\frac{\theta}{2^n}\right)+\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3}+\frac{\phi}{2^n}\right)$$
$$+\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3}+\frac{\psi}{2^n}\right)$$

= the product of the same three tangents.

22. If $\tan\frac{1}{2}(y+z)+\tan\frac{1}{2}(z+x)+\tan\frac{1}{2}(x+y)=0$, prove that $\sin x+\sin y+\sin z+3\sin(x+y+z)=0$. [*C. H. 1935.*]

23. If $\dfrac{\cos(B+C)}{\cos A}+\dfrac{\cos(C+A)}{\cos B}=\dfrac{2\cos(A+B)}{\cos C}$, prove that

$\tan A$, $\tan C$, $\tan B$ are in A. P., or $A+B+C=n\pi$.

24. If $\dfrac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\dfrac{\sin x+\sin y+\sin z}{\sin(x+y+z)}$,

prove that $\Sigma\sin(y+z)=0$

and that each fraction $=\Sigma\cos(y+z)$. [*C. P. 1938.*]

25. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, show that

(i) $\cos n\alpha+\cos n\beta+\cos n\gamma$ is equal to zero, unless n is a multiple of 3, in which case it is equal to

$3\cos\frac{1}{3}n(\alpha+\beta+\gamma)$. [*C. H. 1936.*]

(ii) $\cos^3(\theta+\alpha)+\cos^3(\theta+\beta)+\cos^3(\theta+\gamma)$
$-3\cos(\theta+\alpha)\cos(\theta+\beta)\cos(\theta+\gamma)$

vanishes whatever be the value of θ.

26. If $\cos x+\cos y+\cos z+\cos x\cos y\cos z=0$, prove that $\sin^4 x(1+\cos y\cos z)^2=\sin^4 y(1+\cos z\cos x)^2$
$=\sin^4 z(1+\cos x\cos y)^2$.

27. The system of equations

$$\frac{\sin(2x-y-z)}{\cos(2x+y+z)}=\frac{\sin(2y-z-x)}{\cos(2y+z+x)}=\frac{\sin(2z-x-y)}{\cos(2z+x+y)}$$

if x, y, z be unequal and each less than π, is equivalent to the single equation

$$\cos 2(y+z)+\cos 2(z+x)+\cos 2(x+y)=0.$$

28. If $(\sin^2 y - \sin^2 z) \cot x + (\sin^2 z - \sin^2 x) \cot y + (\sin^2 x - \sin^2 y) \cot z = 0$,

then either the difference of two angles or the sum of all three is a multiple of π.

29. Show that if $\alpha + \beta + \gamma + x + y + z = 0$,

then $$\begin{vmatrix} \tan(\alpha + x) & \tan(\beta + x) & \tan(\gamma + x) \\ \tan(\alpha + y) & \tan(\beta + y) & \tan(\gamma + y) \\ \tan(\alpha + z) & \tan(\beta + z) & \tan(\gamma + z) \end{vmatrix} = 0.$$

30. If $\dfrac{\tan(\theta - \alpha)}{p} = \dfrac{\tan(\phi - \alpha)}{q} = \dfrac{\tan(\psi - \alpha)}{r}$,

show that $\Sigma p(q - r)^2 \cot(\phi - \psi) = 0$.

Sec. B—INEQUALITIES

110. The following examples will illustrate some important types of trigonometrical inequalities.

Ex. 1. *If A, B, C are the angles of a triangle, find the maximum values of*

(i) $\sin A + \sin B + \sin C$.

(ii) $\sin A \sin B \sin C$.

Let us suppose that C remains constant, while A and B vary.

(i) $\sin A + \sin B + \sin C = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) + \sin C$

$$= 2 \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(A - B) + \sin C.$$

This expression is a maximum when $A = B$.

Hence, so long as any two of the angles A, B, C are unequal, the expression $\sin A + \sin B + \sin C$ is not a maximum ; that is the expression is maximum when $A = B = C = 60°$.

Hence, the maximum value $= 3 \sin 60° = \dfrac{3\sqrt{3}}{2}$.

(ii) $2 \sin A \sin B \sin C = \{\cos (A-B) - \cos (A+B)\} \sin C$
$$= \{\cos (A-B) + \cos C\} \sin C.$$

This expression is a maximum when $A = B$.

Hence, by reasoning as before, $\sin A \sin B \sin C$ has its maximum value when $A = B = C = 60°$.

Thus, the maximum value $= \sin^3 60° = \frac{3\sqrt{3}}{8}$.

Note. The above example is a particular case of the following more general theorem :

If the sum of n angles, each positive and less than a right angle, is constant, then the sum of the product of the sines of the angle is greatest when the angles are all equal.

The method of proof is similar to that of Ex. 1. So long as any two of the n angles are unequal, the sum or product of the sines can be increased by replacing each of these two angles by their arithmetic mean and hence the sum or the product is greatest when all the angles are equal.

A similar theorem holds for the *cosines* of n angles.

Ex. 2. *Show that* $\frac{\sec^2\theta - \tan\theta}{\sec^2\theta + \tan\theta}$ *lies between* 3 *and* $\frac{1}{3}$. [*C. H. 1938.*]

Let $x = \frac{\sec^2\theta - \tan\theta}{\sec^2\theta + \tan\theta} = \frac{1+\tan^2\theta - \tan\theta}{1+\tan^2\theta + \tan\theta}$.

$\therefore \quad (x-1)\tan^2\theta + (x+1)\tan\theta + (x-1) = 0.$

In order that the value of $\tan\theta$ found from this quadratic may be real, we must have

$$(x+1)^2 > 4(x-1)^2,$$

i.e., $-3x^2 + 10x - 3$ must be positive,

i.e., $(3x-1)(x-3)$ must be negative.

Hence, x lies between 3 and $\frac{1}{3}$.

Ex. 3. *Show that if θ lies between* 0 *and* π,

$$\cot \tfrac{1}{2}\theta - \cot\theta > 2. \qquad [\text{ C. P. 1934. }]$$

$$\cot \tfrac{1}{4}\theta - \cot \theta = \frac{\cos \frac{1}{4}\theta}{\sin \frac{1}{4}\theta} - \frac{\cos \theta}{\sin \theta} = \frac{\sin \theta \cos \frac{1}{4}\theta - \cos \theta \sin \frac{1}{4}\theta}{\sin \theta \sin \frac{1}{4}\theta}$$

$$= \frac{\sin \frac{3}{4}\theta}{\sin \frac{1}{4}\theta \sin \theta} = \frac{3 \sin \frac{1}{4}\theta - 4 \sin^3 \frac{1}{4}\theta}{\sin \frac{1}{4}\theta \sin \theta}$$

$$= \frac{3 - 4 \sin^2 \frac{1}{4}\theta}{\sin \theta} = \frac{1 + 2 \cos \frac{1}{2}\theta}{\sin \theta}$$

$$= \operatorname{cosec} \theta + \frac{2 \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} = \operatorname{cosec} \theta + \operatorname{cosec} \tfrac{1}{2}\theta.$$

Now, since θ lies between 0 and π, cosec θ and cosec $\frac{1}{2}\theta$ are both positive and each not less than 1. Hence the result.

EXAMPLES XV(b)

1. If A, B, C are the angles of a triangle, find the *maximum* values of

(i) $\cos A + \cos B + \cos C$.

(ii) $\cos A \cos B \cos C$. [*C. H. 1937.*]

(iii) $\tan \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}$.

2. If $A + B + C = \pi$, find the *minimum* values of

(i) $\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C$.

(ii) $\sec A + \sec B + \sec C$.

(iii) $\tan A + \tan B + \tan C$.

(iv) $\cot A + \cot B + \cot C$.

(v) $\tan^2 \dfrac{A}{2} + \tan^2 \dfrac{B}{2} + \tan^2 \dfrac{C}{2}$.

[*Use* $\Sigma \tan \dfrac{B}{2} \tan \dfrac{C}{2} = 1$.]

(vi) $\cot^2 A + \cot^2 B + \cot^2 C$. [*Use* $\Sigma \cot B \cot C = 1$.]

(vii) $\sin^2 \dfrac{A}{2} + \sin^2 \dfrac{B}{2} + \sin^2 \dfrac{C}{2}$.

[*Use* $\Sigma \sin^2 \dfrac{A}{2} = \Sigma \frac{1}{2}(1 - \cos A)$.]

3. Write down the general theorems corresponding to the inequalities in Ex. 2(i), (ii), (iii), (iv).

4. If A, B, C are the angles of a triangle, show that $2(\cot A + \cot B + \cot C)$ is not less than $(\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C)$. [*C. P. 1933.*]

5. If A, B, C be the angles of a triangle, show that $\cos A + \cos B + \cos C$ lies between 1 and $\frac{3}{2}$. [*C. H. 1936.*]

6. Prove that the maximum triangle which can be inscribed in a given circle is equilateral.

[*Area* $= \frac{1}{2}bc \sin A = 2R^2 \sin A \sin B \sin C$.]

7. Prove that the maximum triangle whose perimeter is given is equilateral.

[$\Delta = s^2 \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C$; *use Ex. 1(iii).*]

8. In any triangle, show that the in-radius is never greater than half the circum-radius.

[*Use* $r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.]

9. Prove that in any plane triangle, the value of $\tan B \tan C + \tan C \tan A + \tan A \tan B$ cannot lie between 0 and 9.

[*The exp.* $= 1 + \sec A \sec B \sec C$.]

10. Show that the geometric mean of the cosines of n acute angles is never greater than the cosines of the arithmetic mean of the angles.

11. Show that $\tan 3A \cot A$ cannot lie between 3 and $\frac{1}{3}$.

12. Find the maximum value of $\dfrac{\tan^2 x - \cot^2 x + 1}{\tan^2 x + \cot^2 x - 1}$.

13. Show that in a triangle ABC,

$$8 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \not> 1.$$

[$OI^2 = R^2 (1 - 8 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.]

14. Show that in a triangle

$x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C$ is positive unless $x:y:z=a:b:c$.

15. If
$$\tan\theta+\tan\phi=a,$$
$$\sec\theta+\sec\phi=b,$$
$$\operatorname{cosec}\theta+\operatorname{cosec}\phi=c,$$
and if b and c are of the same sign, show that $bc > 2a$.

ANSWERS

1. (i) $\frac{3}{2}$. (ii) $\frac{1}{8}$. (iii) $\frac{1}{3\sqrt{3}}$.

2. (i) $2\sqrt{3}$. (ii) 6. (iii) $3\sqrt{3}$. (iv) $\sqrt{3}$.
(v) 1. (vi) 1. (vii) $\frac{3}{4}$.

12. $\frac{4}{3}$.

Sec. C.—ELIMINATION

111. The elimination of variables from given trigonometrical equations is a very important and common mathematical problem. There are no set rules to effect the elimination. The form of the equations will often suggest special methods and in addition to the usual algebraical artifices, we shall always have at our disposal the identical relations subsisting among the trigonometrical functions.

The following examples will illustrate some useful methods of elimination.

Ex. 1. *Eliminate θ between the equations*
$$a\cos\theta+b\sin\theta+c=0$$
$$a'\cos\theta+b'\sin\theta+c'=0.$$

From the given equations, we have, by cross-multiplication,
$$\frac{\cos\theta}{bc'-b'c}=\frac{\sin\theta}{ca'-c'a}=\frac{1}{ab'-a'b}.$$
$$\therefore\quad \cos\theta=\frac{bc'-b'c}{ab'-a'b};\ \text{and}\ \sin\theta=\frac{ca'-c'a}{ab'-a'b}.$$

Squaring and adding, we get,

$$(bc'-b'c)^2+(ca'-c'a)^2=(ab'-a'b)^2$$

as the required eliminant.

Ex. 2. *Eliminate θ from the equations*

$$x \sin\theta+y\cos\theta=2a\sin 2\theta$$
$$x\cos\theta-y\sin\theta=a\cos 2\theta.$$

[*C. H. 1929, '31.*]

Solving as simultaneous equations in x and y, we have

$$\begin{aligned} x&=a(\cos 2\theta\cos\theta+2\sin 2\theta\sin\theta)\\ &=a[\cos(2\theta-\theta)+\sin 2\theta\sin\theta]\\ &=a(\cos\theta+2\sin^2\theta\cos\theta),\\ y&=a(2\sin 2\theta\cos\theta-\cos 2\theta\sin\theta)\\ &=a(\sin\theta+\sin 2\theta\cos\theta)=a(\sin\theta+2\sin\theta\cos^2\theta). \end{aligned}$$

$$\begin{aligned} \therefore\quad x+y&=a(\sin\theta+\cos\theta)(1+2\sin\theta\cos\theta)\\ &=a(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)^2=a(\cos\theta+\sin\theta)^3. \end{aligned}$$

Similarly, $x-y=a(\cos\theta-\sin\theta)(1-2\sin\theta\cos\theta)=a(\cos\theta-\sin\theta)^3.$

$$\therefore\quad a^{\frac{1}{3}}(\cos\theta+\sin\theta)=(x+y)^{\frac{1}{3}} \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

$$a^{\frac{1}{3}}(\cos\theta-\sin\theta)=(x-y)^{\frac{1}{3}} \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

Hence, squaring and adding (i) and (ii), we have

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}},$$

as the required eliminant.

Ex. 3. *Eliminate θ from the equations*

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$$

$$x\sin\theta-y\cos\theta=(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}.$$

[*C. H. 1934.*]

Squaring the given equations, we have

$$\begin{aligned} \frac{x^2}{a^2}\cos^2\theta+\frac{2xy}{ab}\sin\theta\cos\theta+\frac{y^2}{b^2}\sin^2\theta&=1\\ &=\sin^2\theta+\cos^2\theta. \quad \cdots \quad \cdots \quad (1) \end{aligned}$$

$$\begin{aligned} x^2\sin^2\theta-2xy\sin\theta\cos\theta+y^2\cos^2\theta\\ =a^2\sin^2\theta+b^2\cos^2\theta. \quad \cdots \quad \cdots \quad (2) \end{aligned}$$

Transposing, (1) reduces to

$$\frac{x^2-a^2}{a^2}\cos^2\theta+\frac{2xy}{ab}\sin\theta\cos\theta+\frac{y^2-b^2}{b^2}\sin^2\theta=0. \quad \cdots \quad (3)$$

Transposing and dividing by ab, (2) reduces to

$$\frac{x^2-a^2}{ab}\sin^2\theta-\frac{2xy}{ab}\sin\theta\cos\theta+\frac{y^2-b^2}{ab}\cos^2\theta=0. \quad \cdots \quad (4)$$

Adding (3) and (4), we get,

$$\frac{x^2-a^2}{a}\left(\frac{\cos^2\theta}{a}+\frac{\sin^2\theta}{b}\right)+\frac{y^2-b^2}{b}\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)=0,$$

or, $$\left[\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right]\left[\frac{x^2-a^2}{a}+\frac{y^2-b^2}{b}\right]=0,$$

$$\therefore \quad \frac{x^2-a^2}{a}+\frac{y^2-b^2}{b}=0,$$

i.e., $$\frac{x^2}{a}+\frac{y^2}{b}=a+b.$$

This is the required eliminant.

Ex. 4. *Show that the equations*

$$a\cos\phi\cos\psi+b\sin\phi\sin\psi=c,$$

$$a\cos\psi\cos\theta+b\sin\psi\sin\theta=c,$$

$$a\cos\theta\cos\phi+b\sin\theta\sin\phi=c,$$

are inconsistent, unless $bc+ca+ab=0$.

From the last two equations we see that ϕ and ψ satisfy the equation in x, namely

$$a\cos\theta\cos x+b\sin\theta\sin x=c.$$

Writing t for $\tan\frac{1}{2}x$, $\cos x=\frac{1-t^2}{1+t^2}$, $\sin x=\frac{2t}{1+t^2}$, and simplifying, we therefore get $\tan\frac{1}{2}\phi$ and $\tan\frac{1}{2}\psi$ to be the roots of the equation,

$$(c+a\cos\theta)\,t^2-2b\sin\theta.t+(c-a\cos\theta)=0.$$

Thus, $$\tan\tfrac{1}{2}\phi+\tan\tfrac{1}{2}\psi=\frac{2b\sin\theta}{c+a\cos\theta}$$

$$\tan\tfrac{1}{2}\phi\tan\tfrac{1}{2}\psi=\frac{c-a\cos\theta}{c+a\cos\theta}.$$

$$\therefore \quad \tan\tfrac{1}{2}(\phi+\psi)=\frac{\tan\frac{1}{2}\phi+\tan\frac{1}{2}\psi}{1-\tan\frac{1}{2}\phi\tan\frac{1}{2}\psi}=\frac{b}{a}\tan\theta.$$

Similarly, $\tan \frac{1}{2}(\psi+\theta)=\dfrac{b}{a}\tan\phi$.

$$\therefore \quad \tan \tfrac{1}{2}(\phi-\theta)=\tan\{\tfrac{1}{2}(\phi+\psi)-\tfrac{1}{2}(\psi+\theta)\}$$

$$=\frac{\dfrac{b}{a}(\tan\theta-\tan\phi)}{1+\dfrac{b^2}{a^2}\tan\theta\tan\phi}$$

$$=\frac{ab\sin(\theta-\phi)}{a^2\cos\theta\cos\phi+b^2\sin\theta\sin\phi}$$

whence, (remembering that θ and ϕ are different, otherwise two of the given equations become identical)

$$a^2\cos\theta\cos\phi+b^2\sin\theta\sin\phi=-ab\{1+\cos(\phi-\theta)\},$$

or, $\quad a(a+b)\cos\theta\cos\phi+b(a+b)\sin\theta\sin\phi+ab=0,$

or, $\quad$ using the last of the given equations,

$$(a+b)c+ab=0,$$

i.e., $\quad bc+ca+ab=0.$

Note. A system of equations which is inconsistent unless the coefficients satisfy a certain relation, is said to be *prismatic*. When this relation is satisfied, the equations have an infinite number of solutions ; in fact in that case any one equation can be deduced from the others.

EXAMPLES XV(c)

1. Eliminate θ from the following pair of equations :

(i) $\cot\theta(1+\sin\theta)=4a$

$\cot\theta(1-\sin\theta)=4b$.

(ii) $x=a\cos\theta+b\cos 2\theta$

$y=a\sin\theta+b\sin 2\theta$.

(iii) $x=\tan\theta+\tan 2\theta$

$y=\cot\theta+\cot 2\theta$.

(iv) $a \sin\theta + b\cos\theta = 1$

$a \operatorname{cosec}\theta - b\sec\theta = 1.$

(v) $x = \sin\theta + \cos\theta\sin 2\theta$

$y = \cos\theta + \sin\theta\sin 2\theta.$

(vi) $x + a = a(2\cos\theta - \cos 2\theta)$

$y = a(2\sin\theta - \sin 2\theta).$

(vii) $x = 3\sin\theta - \sin 3\theta$

$y = \cos 3\theta + 3\cos\theta,$

(viii) $x = \cot\theta + \tan\theta$

$y = \sec\theta - \cos\theta.$

(ix) $x\sin\theta - y\cos\theta = \sqrt{x^2 + y^2}$

$\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2} = \dfrac{1}{x^2 + y^2}.$

(x) $\dfrac{x}{a} = \cos\theta + \cos 2\theta$

$\dfrac{y}{b} = \sin\theta + \sin 2\theta.$

(xi) $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

$\dfrac{ax\sin\theta}{\cos^2\theta} + \dfrac{by\cos\theta}{\sin^2\theta} = 0.$

(xii) $\dfrac{x}{a}\cos\theta - \dfrac{y}{b}\sin\theta = \cos 2\theta$

$\dfrac{x}{a}\sin\theta + \dfrac{y}{b}\cos\theta = 2\sin 2\theta.$

2. If $\cos(\theta - \alpha) = a$, and $\sin(\theta - \beta) = b$, show that

$$a^2 - 2ab\sin(\alpha - \beta) + b^2 = \cos^2(\alpha - \beta).$$

3. Eliminate θ and ϕ from the equations :

(i) $$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

$$\frac{\cos\theta\cos\phi}{a^2}+\frac{\sin\theta\sin\phi}{b^2}=0.$$ [*C. H. 1939, '43.*]

(ii) $$\sin\theta+\sin\phi=a$$

$$\cos\theta+\cos\phi=b$$

$$\sin 2\theta+\sin 2\phi=2c.$$ [*C. H. 1938.*]

4. If α, β, γ be unequal angles each less than 2π, prove that the equations

$$\cos(\theta+\alpha)\sec 2\alpha=\cos(\theta+\beta)\sec 2\beta=\cos(\theta+\gamma)\sec 2\gamma$$

cannot co-exist unless

$$\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)=0.$$ [*C. H. 1932, '40.*]

5. Prove that the equations

$$a^3\tan\theta+a^2\tan\theta\cot\phi+\cot\phi=0$$
$$b^3\tan\theta+b^2\tan\theta\cot\phi+\cot\phi=0$$
$$c^3\tan\theta+c^2\tan\theta\cot\phi+\cot\phi=0$$

can co-exist if either $bc+ca+ab=0$, or else two of the constants a, b, c are equal.

6. If $a\cos\alpha+b\cos\beta+c\cos\gamma=0$

$$a\sin\alpha+b\sin\beta+c\sin\gamma=0$$
$$a\sec\alpha+b\sec\beta+c\sec\gamma=0,$$

prove that, in general, $\pm a\pm b\pm c=0.$

7. Eliminate θ and ϕ from the equations

$$\sin\theta+\sin\phi=a$$
$$\cos\theta+\cos\phi=b$$
$$\tan\theta+\tan\phi=c.$$

8. Eliminate x and y from the equations

$$\cos x + \cos y = a$$
$$\cos 2x + \cos 2y = b$$
$$\cos 3x + \cos 3y = c. \qquad [\,C.\ P.\ 1941.\,]$$

9. If $\quad a \sin\theta - b\cos\theta = -\sin 4\theta$

and $\quad a\cos\theta + b\sin\theta = \frac{5}{2} - \frac{3}{2}\cos 4\theta,$

then $(a+b)^{\frac{2}{3}} + (a-b)^{\frac{2}{3}} = 2.$

10. Eliminate x and y from

$$\frac{a\sin^2 x + b\sin^2 y}{b\cos^2 x + c\cos^2 y} = \frac{b\sin^2 x + c\sin^2 y}{c\cos^2 x + a\cos^2 y} = \frac{c\sin^2 x + a\sin^2 y}{a\cos^2 x + b\cos^2 y}.$$

11. Eliminate θ and ϕ from the equations

(i) $\tan\theta + \tan\phi = a$, $\cot\theta + \cot\phi = b$,
$\theta + \phi = a$.

(ii) $x\sin^2\theta + y\cos^2\theta = a$, $x\cos^2\phi + y\sin^2\phi = b$,
$x\tan\theta = y\tan\phi$. [*C. P. 1942.*]

12. If $\dfrac{\cos(a-3\theta)}{\cos^3\theta} = \dfrac{\sin(a-3\theta)}{\sin^3\theta} = m$, show that

$$m^2 + m\cos a = 2. \qquad [\,C.\ H.\ 1957.\,]$$

13. Eliminate a, b, c from

$$b\cos\gamma + c\cos\beta = a$$
$$c\cos a + a\cos\gamma = b$$
$$a\cos\beta + b\cos a = c.$$

14. Eliminate a, β, γ from

$$\cos a + \cos\beta + \cos\gamma = a$$
$$\sin a + \sin\beta + \sin\gamma = b$$
$$\cos 2a + \cos 2\beta + \cos 2\gamma =$$
$$\sin 2a + \sin 2\beta + \sin 2\gamma = c$$

15. Show that the equations

$$a \cos(\phi+\psi) + b \sin(\phi-\psi) + c = 0,$$
$$a \cos(\psi+\theta) + b \sin(\psi-\theta) + c = 0,$$
$$a \cos(\theta+\phi) + b \sin(\theta-\phi) + c = 0,$$

cannot hold simultaneously for unequal values of θ, ϕ, ψ unless $a^2 - b^2 + 2bc = 0$.

[*Replace a by $a+b$, b by $b-a$ and c by $-c$ in Ex. 4 worked out.*]

16. If $(x-a)\cos\theta + y\sin\theta = (x-a)\cos\phi + y\sin\phi = a$

and $$\tan\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\phi = 2e,$$

and θ and ϕ are unequal angles less than 360°, prove that $y^2 = 2ax - (1-e^2)x^2$. [*C. H. 1937.*]

ANSWERS

1. (i) $(a^2-b^2)^2 = ab$. (ii) $a^2\{(x+b)^2+y^2\} = (x^2+y^2-b^2)^2$.

(iii) $(x+3y)^2 = xy^2(x+2y)$. (iv) $a^2+b^2 = 1+b^{\frac{2}{3}}-b^{\frac{4}{3}}$.

(v) $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}} = 2$. (vi) $(x^2+y^2+2ax)^2 = 4a^2(x^2+y^2)$.

(vii) $x^{\frac{2}{3}}+y^{\frac{2}{3}} = 4^{\frac{2}{3}}$. (viii) $x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}} = 1$.

(ix) $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2} = 1$. (x) $\dfrac{2x}{a} = \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-3\right)$.

(xi) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.

(xii) $\left(\dfrac{x}{a}+\dfrac{y}{b}\right)^{\frac{2}{3}}+\left(\dfrac{x}{a}-\dfrac{y}{b}\right)^{\frac{2}{3}} = 2$.

3. (i) $x^2+y^2 = a^2+b^2$. (ii) $(ab-c)(a^2+b^2) = 2ab$.

7. $(a^2+b^2)^2 - 4a^2 = \dfrac{8ab}{c}$. **8.** $a(3+8b-2a^2) = c$.

10. $a^3+b^3+c^3-3abc = 0$.

11. (i) $ab = (b-a)\tan\alpha$. (ii) $x^2(y-a)(y-b) = y^2(x-a)(x-b)$.

13. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma = 1$.

14. $(a^2+b^2)(a^2+b^2-4)+c^2+d^2 = 2c(a^2-b^2)+4abd$.

Sec. D.—SYMMETRIC FUNCTIONS OF ROOTS OF EQUATIONS

112. Functions of trigonometrical ratios of angles of the type $\left(a+\frac{r\pi}{n}\right)$ or $\frac{r\pi}{n}$ can be deduced from the expansions for $\sin n\theta$, $\cos n\theta$, (*Arts. 94 and 95*) and the expansion for $\tan n\theta$ (*Art. 52*).

From Art. 94, we have, if n be *odd*,

$$(2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{2\,!}(2\cos\theta)^{n-4} - $$
$$+(-1)^{\frac{n-1}{2}}.n.2\cos\theta = 2\cos n\theta.$$

If θ has any one of the n values a, $a+\frac{2\pi}{n}$, $\cdots$, $a+\frac{2(n-1)\pi}{n}$,

then, $\cos n\theta$ has the value $\cos na$.

Hence, putting $x=2\cos\theta$, the equation

$$x^n - nx^{n-2} + \frac{n(n-3)}{2\,!}x^{n-4} - \cdots\cdots$$
$$\cdots+(-1)^{\frac{n-1}{2}}nx - 2\cos na = 0 \qquad \cdots \quad (1)$$

has the roots

$$2\cos a,\ 2\cos\left(a+\frac{2\pi}{n}\right),\ \cdots\cdots,\ 2\cos\left\{a+\frac{2(n-1)\pi}{n}\right\}.$$

Similarly, if n be *even*,

$$x^n - nx^{n-2} + \frac{n(n-3)}{2\,!}x^{n-4} - \cdots + (-1)^{\frac{n}{2}}.2 - 2\cos na = 0 \cdots (2)$$

has the roots

$$2\cos\left(a+\frac{2r\pi}{n}\right),\ r=0,1,2,\ldots n-1.$$

The expansion of $\sin n\theta$ (*Art. 95*) gives an equation with roots of the same type.

The expansion of $\tan n\theta$ (*Art. 52*) gives an equation with roots of the type $\tan\left(a+\dfrac{r\pi}{n}\right)$.

Again, expansions of Art. 96 can also be similarly employed to obtain symmetric functions of roots.

In these equations making a approach zero, equations involving functions of $r\pi/n$ can be derived. In such cases it is often necessary to use the relation

$$\underset{\theta\to 0}{Lt}\ \frac{\sin n\theta}{\sin\theta}=\underset{\theta\to 0}{Lt}\ \frac{\sin n\theta}{n\theta}\cdot\frac{\theta}{\sin\theta}\cdot n=n.$$

Ex. 1. *If n be even, find the value of*

(*i*) $\cos a\ \cos\left(a+\dfrac{2\pi}{n}\right)\cdots\cdots\cos\left\{a+\dfrac{(n-1)2\pi}{n}\right\}$.

(*ii*) $\sec a+\sec\left(a+\dfrac{2\pi}{n}\right)+\cdots\cdots+\sec\left\{a+\dfrac{(n-1)2\pi}{n}\right\}$.

Let $x_1, x_2, \ldots\ldots x_n$ be the roots of the equation (2) above; then

(i) the given exp. $=\dfrac{1}{2^n}\cdot x_1x_2\cdots\cdots x_n$

$$=\frac{1}{2^n}\left\{\left(-1\right)^{\frac{n}{2}}\cdot 2-2\cos na\right\}$$

$$=\frac{1}{2^{n-1}}\left\{\left(-1\right)^{\frac{n}{2}}-\cos na\right\}.$$

(ii) the given exp. $=2\left\{\dfrac{1}{x_1}+\dfrac{1}{x_2}+\cdots\cdots+\dfrac{1}{x^n}\right\}$

$$=2\,\frac{\Sigma x_2x_3\cdots\cdots x_n}{x_1x_2\cdots\cdots x_n}=2\cdot\frac{-\text{coeff. of } x}{\text{constant term}}$$

$$=\frac{0}{\text{const. term}}=0.$$

Ex. 2. *If n be odd, find the value of*

(*i*) $\tan a+\tan\left(a+\dfrac{\pi}{n}\right)+\tan\left(a+\dfrac{2\pi}{n}\right)+\cdots\cdots+\tan\left\{a+\dfrac{(n-1)\pi}{n}\right\}$

(*ii*) $\tan\dfrac{\pi}{n}\tan\dfrac{2\pi}{n}\cdots\cdots\tan\dfrac{(n-1)\pi}{n}$.

From Art. 52, if n be *odd*,

$$\tan n\theta = \frac{n\tan\theta - {}^nc_3\tan^3\theta + \cdots\cdots + (-1)^{\frac{n-1}{2}}\tan^n\theta}{1 - {}^nc_2\tan^2\theta + \cdots\cdots + (-1)^{\frac{n-1}{2}} n\tan^{n-1}\theta}.$$

If θ has any one of the n values a, $a+\frac{\pi}{n}$, $a+\frac{2\pi}{n}$, ...$a+\frac{(n-1)\pi}{n}$, then $\tan n\theta$ has the value $\tan na$.

Hence, putting $x=\tan\theta$, we have

$$(-1)^{\frac{n-1}{2}} x^n + \cdots\cdots - {}^nc_3x^3 + nx$$
$$- \tan na \{(-1)^{\frac{n-1}{2}} nx^{n-1} + \cdots\cdots - {}^nc_2x^2 + 1\} = 0. \quad \cdots \quad (3)$$

This equation has n roots, $\tan\left(a+\frac{r\pi}{n}\right)$, $r=0, 1, 2, \ldots\ldots n-1$.

(i) Given exp. = sum of all the roots of equation (3)

$$= -\frac{\text{coeff. of } x^{n-1}}{\text{coeff. of } x^n} = n\tan na.$$

(ii) From equation (3), we have

$$\tan a \tan\left(a+\frac{\pi}{n}\right)\tan\left(a+\frac{2\pi}{n}\right)\cdots \text{ to } n \text{ factors} = (-1)^{\frac{n-1}{2}}\tan na.$$

$$\therefore \quad \tan\left(a+\frac{\pi}{n}\right)\tan\left(a+\frac{2\pi}{n}\right)\cdots\tan\left\{a+\frac{(n-1)\pi}{n}\right\}$$
$$= (-1)^{\frac{n-1}{2}}\frac{\tan na}{\tan a}.$$

Now, making a approach zero, we get

$$\tan\frac{\pi}{n}\tan\frac{2\pi}{n}\cdots\tan\frac{(n-1)\pi}{n} = (-1)^{\frac{n-1}{2}} \underset{a\to 0}{Lt} \frac{\tan na}{\tan a}$$
$$= (-1)^{\frac{n-1}{2}} n.$$

113. The evaluation of symmetric functions of the trigonometrical ratios of angles $\frac{\pi}{n}, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots$ for particular values of n may be obtained in various ways ; most of them

ultimately depend on the formation of an equation of which the given trigonometrical ratios are the roots.

Ex. 3. *Prove that*

(*i*) $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$.

(*ii*) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{8}$.

If $7\theta=2n\pi$, where n is zero or any integer, we have

$$4\theta=2n\pi-3\theta.$$

$$\therefore\quad \cos 4\theta=\cos 3\theta.$$

By giving to n the values 0, 1, 2, 3, the values of $\cos\theta$ obtained from the equation are

$$\cos 0,\ \cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \cos\frac{6\pi}{7}.$$

It can be easily seen that no new values of $\cos\theta$ are found by ascribing to n the values 4, 5, 6,..., for $\cos\frac{8\pi}{7}=\cos\frac{6\pi}{7}$, $\cos\frac{10\pi}{7}=\cos\frac{4\pi}{7}$ etc.

Now, $\quad \cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$,

and $\quad \cos 3\theta=4\cos^3\theta-3\cos\theta$;

therefore, putting $x=\cos\theta$, the equation $\cos 4\theta=\cos 3\theta$ becomes

$$8x^4-8x^2+1=4x^3-3x,$$

or, $\quad 8x^4-4x^3-8x^2+3x+1=0.$

Since, $\cos 0$, $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$ are the roots of the above equation

(i) $\cos 0+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=\frac{4}{8}$.

$\therefore\quad \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$.

(ii) $\cos 0\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{8}$,

or, $\quad \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{8}$.

Note. Since the angles $\frac{2}{7}\pi$, $\frac{4}{7}\pi$ etc. are in A. P. with the common difference $\frac{2}{7}\pi$, the values of the expressions of the type (i) can also be easily obtained by applying the formula for the sum of cosines of n angles in A. P.

114. The symmetric functions of the roots of certain types of equations can be best obtained by expressing sine, cosine and tangent in terms of tangent of half angles. The following example illustrates the method.

Ex. 4. *Prove that the equation*

$$a^2 \cos^2\theta + b^2 \sin^2\theta + 2ga \cos\theta + 2fb \sin\theta + c = 0$$

has 4 roots and the sum of these roots is an even multiple of π.

Let $t = \tan\frac{\theta}{2}$; then $\sin\theta = \frac{2t}{1+t^2}$; $\cos\theta = \frac{1-t^2}{1+t^2}$.

Substituting these values in the given equation and simplifying, we get

$$t^4(a^2 - 2ga + c) + 4fbt^3 + t^2(4b^2 - 2a^2 + 2c) + 4fbt + a^2 + 2ga + c = 0. \quad \cdots \quad (1)$$

This equation has 4 roots, say, t_1, t_2, t_3, t_4.

Now, $\tan\left(\frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4} = 0$, [see *Art. 56.*]

$$\begin{aligned} \text{since, } s_1 - s_3 &= \Sigma \tan\tfrac{1}{2}\theta_1 - \Sigma \tan\tfrac{1}{2}\theta_1 \tan\tfrac{1}{2}\theta_2 \tan\tfrac{1}{2}\theta_3 \\ &= \Sigma t_1 - \Sigma t_1 t_2 t_3 \\ &= \frac{-4fb + 4fb}{a^2 - 2ga + c} \quad \text{[from equation (1)]} \\ &= 0. \end{aligned}$$

$$\therefore \quad \frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} = n\pi, \text{ i.e., } \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2n\pi.$$

Note. This is a solution of the well-known geometrical problem namely "*If a circle and an ellipse intersect in four points, the sum of the eccentric angles of the four points is an even multiple of two right angles.*"

Ex. 5. *If* $\dfrac{\cos(\theta+a)}{\sin^3\theta}=\dfrac{\cos(\phi+a)}{\sin^3\phi}=\dfrac{\cos(\psi+a)}{\sin^3\psi}$, *where no two of the angles* θ, ϕ, ψ *differ by a multiple of* π, *prove that*

(i) $\tan a=\cot\theta+\cot\phi+\cot\psi$.

(ii) $\theta+\phi+\psi=n\pi$.

Put $\dfrac{\cos(\theta+a)}{\sin^3\theta}=k$; then θ, ϕ, ψ are values of x which satisfy the equation, $\dfrac{\cos(x+a)}{\sin^3 x}=k$.

Now, $\dfrac{\cos(x+a)}{\sin^3 x}=\dfrac{\cos x\cos a-\sin x\sin a}{\sin^3 x}=\dfrac{\cot x\cos a-\sin a}{\sin^2 x}$

$$=(\cot x\cos a-\sin a)\operatorname{cosec}^2 x$$

$$=(\cot x\cos a-\sin a)(1+\cot^2 x)$$

$$=\cot^3 x\cos a-\cot^2 x\sin a+\cot x\cos a-\sin a;$$

$\therefore$ $\cot\theta, \cot\phi, \cot\psi$ are the roots of the cubic in λ, namely,

$$\lambda^3\cos a-\lambda^2\sin a+\lambda\cos a-(\sin a+k)=0.$$

$\therefore$ $\Sigma\cot\theta=\tan a$,

and $\Sigma\cot\theta\cot\phi=1$; $\therefore$ $\Sigma\tan\theta=\tan\theta\tan\phi\tan\psi$;

$\therefore$ $\tan(\theta+\phi+\psi)=0$; $\therefore$ $\theta+\phi+\psi=n\pi$.

EXAMPLES XV(d)

1. Prove that if n be *odd*

(i) $\cot a+\cot\left\{a+\dfrac{\pi}{n}\right\}+\cot\left\{a+\dfrac{2\pi}{n}\right\}+\cdots$ to n terms

$=n\cot na$. [*C. H. 1937.*]

(ii) $\sec a\sec\left\{a+\dfrac{2\pi}{n}\right\}\cdots\sec\left\{a+\dfrac{(n-1)2\pi}{n}\right\}$

$=2^{n-1}\sec na$.

(iii) $\sec^2 a+\sec^2\left\{a+\dfrac{2\pi}{n}\right\}+\cdots+\sec^2\left\{a+\dfrac{(n-1)2\pi}{n}\right\}$

$=n^2\sec^2 na$.

2. Prove that the values of x which satisfy the equation

$$1 - nx - \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \cdots + (-1)^{\frac{1}{2}n(n+1)}x^n = 0$$

are given by $x = \tan\dfrac{(4r+1)}{4n}\pi$, where r is any integer.

[*C. H. 1938.*]

3. Show that the roots of the equation

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$$

are $2\cos\dfrac{2\pi}{11}$, $2\cos\dfrac{4\pi}{11}$, etc. [*C. H. 1932.*]

[*Proceed as in Ex. 3 worked out.*]

4. If $\theta_1, \theta_2, \theta_3, \theta_4$ are four roots (which do not differ from one another by a multiple of π) of the equation

$$a \sec\theta + b \operatorname{cosec}\theta = c,$$

prove that $\Sigma\theta_1 = (2n+1)\pi$.

Give a geometrical interpretation of the result.

5. From the identity, $\tan 4\theta = \dfrac{4\tan\theta\,(1-\tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$ calculate the four values of θ which lie between 0 and π and conform to the relation $\tan 4\theta = 1$; and make use of the above formula to exhibit the four roots of the biquadratic

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

as $\tan\dfrac{\pi}{16}$, $\tan\dfrac{5\pi}{16}$, $\tan\dfrac{9\pi}{16}$, $\tan\dfrac{13\pi}{16}$.

6. Prove that

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$$

[*C. P. 1929.*]

[*Compare Note of Art. 113.*]

7. If x_1, x_2, x_3 be the roots of the equation

$$\tan(x+a)=k\tan 2x$$

no two of which differ from one another by a multiple of π, prove that $x_1+x_2+x_3+a$ is a multiple of π.

[*P. H. 1934 ; C. H. 1940.*]

8. Prove that the equation $\cos 2x+a\cos x+b\sin x+c=0$ has in general four solutions x_1, x_2, x_3, x_4 between 0 and 2π. Also prove that $x_1+x_2+x_3+x_4=2n\pi$.

9. Prove that

$$\cot^2 a+\cot^2\left(a+\frac{\pi}{n}\right)+\cot^2\left(a+\frac{2\pi}{n}\right)+\cdots \text{ to } n \text{ terms}$$
$$=n^2\cot^2 na+n(n-1).$$

[*C. H. 1945.*]

10. Prove that

$$\tan\frac{\pi}{4n}\tan\frac{3\pi}{4n}\cdots\cdots\tan\frac{(2n-1)\pi}{4n}=1.$$

11. If x_1, x_2, x_3, x_4 are four different values of x which satisfy the equation

$$a\cos 2x+b\sin 2x-c\cos x-d\sin x+c=0,$$

then, $\Sigma\sin x_1=\dfrac{bc-ad}{a^2+b^2}$.

[*The given equation can be put as a biquadratic in* $\sin x$,

$$\sin^4 x\,(4a^2+4b^2)+\sin^3 x\,(4ad-4bc)+\cdots\cdots=0.\text{]}$$

12. If $\tan\theta, \tan\phi, \tan\psi$ are all different and such that

$$\tan 3\theta=\tan 3\phi=\tan 3\psi$$

then, $\Sigma\tan\theta\,.\,\Sigma\cot\theta=0$.

[$x=\theta, \phi, \psi$ *satisfy* $\tan 3x=k$. $\therefore$ $\tan\theta, \tan\phi, \tan\psi$ *are roots of* $\lambda^3-3k\lambda^2-3\lambda+k=0$.]

13. If $\tan(\alpha-\theta)+\tan(\alpha-\phi)+\tan(\alpha-\psi)=0$

$$\tan(\beta-\theta)+\tan(\beta-\phi)+\tan(\beta-\psi)=0$$

$$\tan(\gamma-\theta)+\tan(\gamma-\phi)+\tan(\gamma-\psi)=0$$

where no two of the angles α, β, γ differ by a multiple of π, prove that

$$\tan(\theta+\phi+\psi)=\tan(\alpha+\beta+\gamma).$$

14. If $a^2\cos\alpha\cos\beta+a(\sin\alpha+\sin\beta)+1=0$

and $a^2\cos\alpha\cos\gamma+a(\sin\alpha+\sin\gamma)+1=0$,

prove that $a^2\cos\beta\cos\gamma+a(\sin\beta+\sin\gamma)+1=0$,

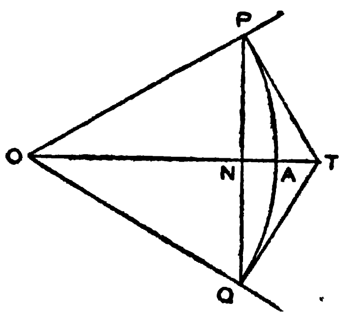
β and γ each being less than π.

Sec. E.—SOME TRIGONOMETRICAL LIMITS

115. *To prove that*

$$\sin\theta < \theta < \tan\theta,$$

where θ is the circular measure of any positive acute angle.



Let AOP be a positive acute angle whose radian measure is θ, and let QOA be an equal angle on the other side of OA. With centre O and any radius, a circle is drawn cutting OP, OA, OQ at P, A, Q respectively. PQ is joined, cutting OA at N. The triangles OPN and OQN are easily seen to be congruent, so that $PN=QN$ and PNQ is

perpendicular to OA. The tangent PT to the circle at P cutting OA at T, $\angle OPT$ is a right angle. TQ being joined, the triangles OPT and OQT are easily proved to be congruent, so that $TP = TQ$.

The figure is thus symmetrical about OA.

Then, from the figure,

$$\sin\theta = \frac{PN}{OP} = \frac{1}{2}\frac{PQ}{OP}$$

$$\theta = \frac{\text{arc } PA}{OP} = \frac{1}{2}\frac{\text{arc } PAQ}{OP}$$

$$\tan\theta = \frac{1}{2}\frac{PT+QT}{OP}.$$

Now, we may take it as axiomatic that the straight line PQ is less than the curved arc PAQ, and that the curved arc PAQ which always bends the same way, being within the triangle PTQ, is less than $PT + QT$.

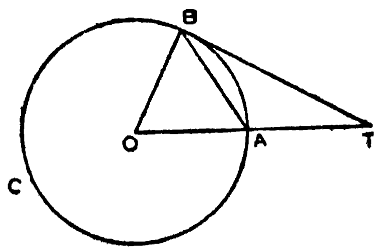
Hence, since $PQ <$ arc $PAQ < PT + QT$,

we have, on dividing throughout by $2OP$

$$\sin\theta < \theta < \tan\theta.$$

Alternative method :

Let ABC be a circle whose centre is O and radius is r.



Let $AOB = \theta$ radians.

Draw BT the tangent at B to meet OA produced at T; then $BT = r\tan\theta$.

We know that if the angle of a sector of a circle of radius r is θ radians, its area $= \frac{1}{2}r^2\theta$.

Now, from the figure, it is obvious that

$$\triangle OAB < \text{sector } OAB < \triangle OTB.$$

$$\therefore \quad \tfrac{1}{2}r^2 \sin\theta < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r.r \tan\theta,$$

$$i.e., \quad \sin\theta < \theta < \tan\theta.$$

116. *To prove that*

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta} = 1$$

$$\underset{\theta\to 0}{\mathrm{Lt}} \cos\theta = 1$$

and $\underset{\theta\to 0}{\mathrm{Lt}} \dfrac{\tan\theta}{\theta} = 1.$

Since, $\sin\theta < \theta < \tan\theta$, we get, on dividing by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

This is true, however small θ may be, provided it is positive. When θ becomes infinitely small, OP and ON practically come into coincidence, so that

$$\cos\theta = \frac{ON}{OP} \text{ ultimately becomes 1.}$$

Hence, $\underset{\theta\to 0}{Lt} \cos\theta = 1.$

In that case $\dfrac{1}{\cos\theta}$ also tends to the value 1. But $\dfrac{\theta}{\sin\theta}$ always lies between 1 and $\dfrac{1}{\cos\theta}$, which ultimately come into coincidence, and so $\dfrac{\theta}{\sin\theta}$ also ultimately coincides with 1.

Thus, $\dfrac{\sin\theta}{\theta} = 1$ in the limit.

Again, from $\sin\theta < \theta < \tan\theta$,

we get by dividing by $\tan\theta$, $\cos\theta < \dfrac{\theta}{\tan\theta} < 1$

and as $\theta \to 0$, $\cos\theta \to 1$ and $\dfrac{\theta}{\tan\theta}$ always lying between $\cos\theta$ and 1 which come into coincidence, $\dfrac{\theta}{\tan\theta}=1$ in the limit and so $\underset{\theta\to 0}{Lt}\dfrac{\tan\theta}{\theta}=1$. Hence the results.

Note 1. The above relations also hold good when θ is negative and $\to 0$. For writing $\theta=-\phi$, ϕ is positive and $\to 0$.

Thus, $\underset{\theta\to 0}{Lt}\dfrac{\sin\theta}{\theta}=\underset{\phi\to 0}{Lt}\dfrac{\sin\phi}{\phi}=1$, etc.

Note 2. The above relations are often expressed loosely but conveniently, by saying that, when θ is a very small angle we can take $\sin\theta=\theta=\tan\theta$ and $\cos\theta=1$. But it should be remembered that the statement is only approximately true, the error becoming less and less as θ grows smaller and smaller. For instance,

$5^\circ=\cdot08725$ radians ; and from the tables we get

$\sin 5^\circ=\cdot08716$ and $\tan 5^\circ=\cdot08749$.

Note 3. It should be carefully noted that the above relations hold good when and only when the angle is expressed in radian measure ; but if any other system of measurement is used, the results require modification, as shown in Ex. 1 below.

Ex. 1. *Find the value of* $\underset{x\to 0}{Lt}\left(\dfrac{\sin x^\circ}{x}\right)$.

Let θ be the number of radians in x°, then

$$\frac{x}{180}=\frac{\theta}{\pi}. \quad \therefore\ x=\frac{180\theta}{\pi} \text{ also } \sin x^\circ=\sin\theta.$$

$$\therefore\ \frac{\sin x^\circ}{x}=\frac{\pi\sin\theta}{180\theta}=\frac{\pi}{180}.\frac{\sin\theta}{\theta}.$$

Now, when $x\to 0$, θ also tends to zero.

$$\therefore\ \underset{x\to 0}{Lt}\left(\frac{\sin x^\circ}{x}\right)=\frac{\pi}{180}\underset{\theta\to 0}{Lt}\left(\frac{\sin\theta}{\theta}\right)=\frac{\pi}{180}.$$

Ex. 2. *Find to 5 places of decimals the value of sin 2° without the use of tables.*

Since, $2^\circ=\dfrac{\pi}{90}$ radians which is small,

$$\therefore\ \sin 2^\circ=\sin\frac{\pi}{90}=\frac{\pi}{90}\text{ (approximately)}=\frac{3\cdot14159}{90}=\cdot03491.$$

Ex. 3. *If θ be the number of radians in an acute angle, show that*

$$\cos\theta > 1-\tfrac{1}{2}\theta^2 \text{ and } \sin\theta > \theta-\tfrac{1}{4}\theta^3.$$

Since, $\cos\theta = 1-2\sin^2\tfrac{1}{2}\theta$ and $\sin\tfrac{1}{2}\theta < \tfrac{1}{2}\theta$,

$$\therefore \quad \cos\theta > 1-2.(\tfrac{1}{2}\theta)^2, \text{ i.e., } > 1-\tfrac{1}{2}\theta^2.$$

Again, $\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = 2\tan\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta$.

$$\begin{aligned}
\therefore \quad \sin\theta &> 2.\tfrac{1}{2}\theta.\cos^2\tfrac{1}{2}\theta && [\because \tan\tfrac{1}{2}\theta > \tfrac{1}{2}\theta]\\
&> \theta(1-\sin^2\tfrac{1}{2}\theta)\\
&> \theta\{1-(\tfrac{1}{2}\theta)^2\} && [\because \sin\tfrac{1}{2}\theta < \tfrac{1}{2}\theta]\\
&> \theta-\tfrac{1}{4}\theta^3.
\end{aligned}$$

Ex. 4. *Show that* $\frac{\sin\theta}{\theta}$ *continually diminishes from* 1 *to* $\frac{2}{\pi}$ *as* θ *continually increases from* 0 *to* $\frac{\pi}{2}$.

Let x denote the radian measure of a small positive angle.

Then, $$\frac{\sin\theta}{\theta} > \frac{\sin(\theta+x)}{\theta+x},$$

if $(\theta+x)\sin\theta > \theta(\sin\theta\cos x+\cos\theta\sin x)$,

i.e., if $\theta\sin\theta(1-\cos x)+(x\sin\theta-\theta\cos\theta\sin x)$ is positive. ... (1)

Now, $1-\cos x$ is positive.

Again, since $\tan\theta > \theta$, $\therefore$ $\sin\theta > \theta\cos\theta$

and since x is an acute angle, $\therefore$ $x > \sin x$.

$\therefore$ $x\sin\theta > \theta\cos\theta\sin x$.

Thus, (1) is positive, and so, $\dfrac{\sin\theta}{\theta} > \dfrac{\sin(\theta+x)}{\theta+x}$.

$\therefore$ $\dfrac{\sin\theta}{\theta}$ continually diminishes as θ continually increases.

When $\theta\to 0$, $\dfrac{\sin\theta}{\theta}\to 1$; when $\theta = \dfrac{\pi}{2}$, $\dfrac{\sin\theta}{\theta} = \dfrac{2}{\pi}$.

Hence, the proposition.

117. *To prove that*

$$\text{(i)} \underset{n\to\infty}{\text{Lt}} \left(\cos\frac{\theta}{n}\right)^n = 1\,; \quad \text{(ii)} \underset{n\to\infty}{\text{Lt}} \left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^n = 1.$$

(i) Let u denote the given expression.

Since, $\cos\frac{\theta}{n} = \left(1 - \sin^2\frac{\theta}{n}\right)^{\frac{1}{2}}$,

$$\therefore \quad \left(\cos\frac{\theta}{n}\right)^n = \left(1 - \sin^2\frac{\theta}{n}\right)^{\frac{1}{2}n}.$$

$$\therefore \quad \log u = \frac{n}{2}\log\left(1 - \sin^2\frac{\theta}{n}\right)$$

$$= \frac{n}{2}\log(1-x), \text{ putting } x = \sin^2\frac{\theta}{n}$$

$$= \frac{n}{2}x\frac{\log(1-x)}{x}.$$

When $n \to \infty$, $\frac{\theta}{n}$ and hence $\sin^2\frac{\theta}{n} \to 0$ and it is known from Algebra that $\underset{x\to 0}{Lt}\frac{\log(1-x)}{x} = -1$.

$$\therefore \quad Lt \log u = Lt\left(-\frac{n}{2}x\right) = -Lt\,\frac{n}{2}\sin^2\frac{\theta}{n} = 0.$$

$$\left[\text{Since, } Lt\,\frac{n}{2}\sin^2\frac{\theta}{n} = Lt\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^2 \times \frac{\theta^2}{2n} = 1 \times 0 = 0.\right].$$

$$\therefore \quad Lt\, u = 1,$$

i.e., the limit of the given expression is 1.

(ii) By Art. 113,

$$\sin\frac{\theta}{n} < \frac{\theta}{n} < \tan\frac{\theta}{n}.$$

Now, dividing throughout by $\sin\frac{\theta}{n}$, and raising all the quantities to the nth power, and then taking their reciprocals, it follows that

$$\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^n \text{ lies between 1 and } \left(\cos\frac{\theta}{n}\right)^n.$$

But by (i), $Lt\left(\cos\frac{\theta}{n}\right)^n = 1.$

Hence, the limit of the given expression is 1.

EXAMPLES XV(e)

1. Show that $\underset{n\to\infty}{Lt}\ n\sin\frac{\theta}{n} = \underset{n\to\infty}{Lt}\ n\tan\frac{\theta}{n} = \theta$

2. Show that $\underset{\theta\to0}{Lt}\frac{\sin a\theta}{\sin b\theta} = \underset{\theta\to0}{Lt}\frac{\tan a\theta}{\tan b\theta} = \frac{a}{b}.$

3. Show that $\underset{x\to0}{Lt}\left(\frac{\tan x^\circ}{x}\right) = \frac{\pi}{180}.$

4. Show that $\underset{x\to0}{Lt}\left(\frac{\sin x'}{x}\right) = \underset{x\to0}{Lt}\left(\frac{\tan x'}{x}\right) = \frac{\pi}{10800}.$

5. Show that $\sin(\cos\theta) < \cos(\sin\theta)$ where θ is a positive acute angle. [*C. H. 1939.*]

6. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$, if $0 < x < \frac{1}{2}\pi$.

7. Show that $2(1-\cos x) > x\sin x$, if $0 < x < \pi$.

8. Show that $\left|\frac{\sinh x}{x}\right| > \left|\frac{\sin x}{x}\right|.$

9. Prove that $\theta > \frac{3\sin\theta}{2+\cos\theta}$, if $0 < \theta < \frac{1}{2}\pi$.

10. If a lies between 0 and $\frac{1}{2}\pi$, show that as θ increases from 0 to a, the expression $a\sin\theta - \theta\sin a$ first continually increases and then continually decreases. [*C. H. 1941.*]

APPENDIX

A SHORT NOTE ON THE CONVERGENCY OF INFINITE SERIES AND PRODUCTS

1. Series of real terms.

Consider the infinite series $u_1 + u_2 + u_3 + \cdots\cdots + u_n + \cdots$ of which the terms $u_1, u_2, u_3, \ldots\ldots$ are all *real*.

Let $S_n = u_1 + u_2 + \cdots + u_n$.

If S_n tends to a definite finite limit S, as n tends to ∞, then the series is said to be **convergent** and S is called the **sum** of the series.

Let us consider the infinite series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots \text{ to infinity.} \qquad \cdots \quad (1)$$

Here, $\quad S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{2^{n-1}}$

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

$\therefore \quad \underset{n \to \infty}{Lt} S_n = 2$, which is finite.

$\therefore$ the above series is convergent, having 2 for its sum.

If S_n tends to $+\infty$ or $-\infty$, as n tends to infinity, the series is said to be **divergent**.

Let us consider the series

$$1 + 2 + 2^2 + 2^3 + \cdots + \text{ to } \infty, \qquad \cdots \qquad \cdots \quad (2)$$

Here, $\quad S_n = 1 + 2 + 2^2 + \cdots + 2^{n-1}$

$$= \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

$$\underset{n\to\infty}{Lt}\ S_n\,.$$

$\therefore$ the above series is divergent.

A series (consisting of positive and negative terms) which is convergent and would remain convergent when all the terms are taken positively is said to be an **absolutely convergent** series, and the series which is originally convergent but becomes divergent when all the terms are taken positively is said to be a **semi-convergent** series.

The most important characteristic property of a semi-convergent series is that a derangement of the order of the terms affects the sum of the series, whereas in an absolutely convergent series it is not so.

Let us consider the series

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots\cdots \text{ to } \infty. \qquad \cdots \qquad (3)$$

Here, $S_n = \dfrac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = \dfrac{2}{3}\left[1+(-1)^{n+1}\dfrac{1}{2^n}\right]$

$\therefore \underset{n\to\infty}{Lt}\ S_n = \frac{2}{3}.$

$\therefore$ the above series is convergent.

Further, since, the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots \text{ to } \infty$$

is also convergent, as proved above,

$\therefore$ the series (3) is *absolutely convergent*, and it can be shown that any derangement of the order of the terms leaves the series convergent, and its sum unaffected.

Now, let us consider the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\cdots \text{ to } \infty. \qquad \cdots \qquad (4)$$

Since, $\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$ for $x < 1$.

$\therefore$ putting $x = 1$, we get

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\cdots \text{ to } \infty.$$

Thus, the series (4) is convergent.

Now, let us examine the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots\cdots \text{ to } \infty. \qquad \cdots \quad (5)$$

That this series is divergent can be shown as follows :

Here,
$$u_3+u_4 > 2.\tfrac{1}{4} \quad \text{or } \tfrac{1}{2}$$
$$u_5+u_6+u_7+u_8 > 4.\tfrac{1}{8} \quad \text{or } \tfrac{1}{2}$$
$$u_9+\cdots\cdots\cdots+u_{16} > 8.\tfrac{1}{16} \quad \text{or } \tfrac{1}{2} \text{ etc.}$$

$\therefore$ $S_n > 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$ to $m+1$ terms say, where m depends upon n and increases with n.

$\therefore$ $S_n > 1+\frac{1}{2}m.$

$\therefore$ $\underset{n\to\infty}{Lt}\ S_n=\infty.$

$\therefore$ the series (5) is divergent.

Here no derangement of the order of the terms in the series (4) is allowed, for it can be shown that the sum will be altered thereby.

This is illustrated in the following example.

We have,

$$\begin{aligned}\log_e 2 &= 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\cdots\cdots \text{ to } \infty\\ &= 1+\tfrac{1}{2}-2.\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}-2.\tfrac{1}{4}+\tfrac{1}{5}+\cdots\\ &= (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots)-(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots)\\ &= 0, \text{ which is absurd.}\end{aligned}$$

It is not always very convenient to find S_n (*i.e.*, the sum up to n terms) of an infinite series and then to find the limit of S_n in order to ascertain the convergence of a series ; so various methods have been devised to test the convergency of infinite series without obtaining S_n.

The notation $|u_r|$ is used to denote the ***absolute value*** of the term u_r (*i.e.*, the positive value of the rth term irrespective of the + or − sign before it). Thus, if $\Sigma|u_n|$ is convergent, Σu_n is said to be absolutely convergent.

A series which is neither convergent nor divergent, is said to be **oscillatory** ; *e.g.*, $1-1+1-1+1-1+\cdots\cdots$

2. Ratio Test.

The first simplest test used to determine the convergency of an infinite series is the Ratio Test.

The test* is :—

If $\underset{n\to\infty}{Lt} \left| \frac{u_{n+1}}{u_n} \right| < 1$, the series is absolutely convergent

> 1, the series is divergent

$= 1$, the test fails, and other tests must be adopted.

3. Series of Complex terms.

Consider the infinite series $z_1 + z_2 + z_3 + \cdots\cdots + z_n + \cdots\cdots$ of which the terms z_1, z_2, z_3 are all *complex*, and let $z_r = x_r + iy_r$, where x_r and y_r are *real*.

The series Σz_n is said to be **convergent**, if the two series of real terms Σx_n and Σy_n are both convergent, and if Σx_n and Σy_n converge respectively to the values x and y, then Σz_n, *i.e.*, $\Sigma(x_n + iy_n)$ converges to the value $x + iy$, and in that case, $x + iy$ is said to be the **sum** of the series Σz_n. Σz_n is said to be **absolutely convergent**, if Σx_n and Σy_n are both absolutely convergent.

4. The Power series.

If the power series $\Sigma a_n z^n$ (where a is real) is absolutely convergent for any particular value of the modulus of z, say r, then it is absolutely convergent for all values of mod. (z), such that mod. $(z) < r$.

If a power series is convergent for all values of z such that mod. $(z) < r$ and is not convergent for any value of z such that mod. $(z) > r$, then the circle with centre at the origin and of radius r is called the **circle of convergence, and** r, the **radius of convergence.**

---

*For proof see any text-book on Higher Algebra.

Thus, a series is absolutely convergent for all points inside the circle of convergence, and is divergent for all points outside it ; but on the circumference of the circle of convergence, the series may converge either absolutely or conditionally, oscillate, or diverge.

The radius of convergence may be finite, zero, or infinite.

5. Infinite Product.

Let $x_1, x_2, \ldots\ldots x_n \ldots$ be a sequence of quantities formed according to some law, and let P_n denote the product of first n of these quantities, so that

$$P_n = x_1 x_2 \ldots x_n.$$

If P_n tends to some finite limit P (different from zero) as n tends to ∞, then the infinite product is said to be **convergent** and P is said to be the limit or the limiting **value** of the infinite product. In other cases, the product is said to be **divergent.**

The following theorem is of great help in determining the convergence of infinite products :

"If the terms of Σu_n become ultimately infinitely small and have ultimately the same sign, then

$$\Pi(1+u_n), \text{ i.e., } (1+u_1)(1+u_2)(1+u_3)\cdots\cdots$$

is *absolutety convergent* if $\Sigma \mid u_n \mid$ is convergont".

If it so happens that $\Pi(1+u_n)$ is convergent (*i.e.*, tends to a finite limit), whilst the series $\Sigma \mid u_n \mid$ is divergent, then the infinite product is said to be **semi-convergent.** A semi-convergent product has the property similar to that of a semi-convergent series, *viz.*, that a derangement of the order of the factors affects the value of the product.

6. Convergency of some well-known Series and Products.

(1) Exponential series.

(i) When x is *real*,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots \qquad \cdots \quad (1)$$

Here, $$Lt \frac{u_{n+1}}{u_n} = Lt \frac{x^n}{\lfloor n} \div \frac{x^{n-1}}{\lfloor n-1} = Lt$$

$$= 0, \text{ for all values of } x.$$

Hence, the series (1) is *absolutely convergent* for all real values of x;

(ii) when x is *complex*, then also the series (1) is *absolutely convergent.*

(2) Logarithmic series.

(i) When x is *real*,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots$$

Here, $$Lt \left| \frac{u_{n+1}}{u_n} \right| = Lt \left| \frac{(-1)^n x^{n+1}}{n+1} \div \frac{(-1)^{n-1} x^n}{n} \right|$$

$$= Lt \frac{|x|}{1+\frac{1}{n}} =$$

Hence, the series is *absolutely convergent* when $|x| < 1$, *i.e.*, $-1 < x < 1$.

When $x = 1$,

the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \cdots$, and is *semi-convergent.*

(ii) When x is *complex*,

the series is convergent provided mod. $(x) < 1$ and also in the case of mod. $(x) = 1$ except when amp. $(x) = (2n+1)\pi$.

(3) sine and cosine series.

(i) When x is *real*,

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\cdots$$

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots\cdots$$

For the sine series,

$$Lt\left|\frac{u_{n+1}}{u_n}\right| = Lt\left|\frac{(-1)^n . x^{2n+1}}{\lfloor 2n+1} \div \frac{(-1)^{n-1} . x^{2n-1}}{\lfloor 2n-1}\right|$$

$$= Lt\ \frac{x^2}{2n(2n+1)} = 0, \text{ for all values of } x.$$

Hence, the series for sin x is *absolutely convergent* for all values of x.

Similarly, it can be shown that the series for cos x is *absolutely convergent* for all values of x.

(ii) When x is *complex*,

then also the series for sin x and cos x are *absolutely convergent*.

Note. The method of proof employed in establishing the expansions of sin α and cos α as given in Arts. 53 and 54 tacitly assumes that *for all values of r*, the limiting value of the expression

$$\frac{\alpha(\alpha-\theta)(\alpha-2\theta)\cdots(\alpha-\overline{r-1}\theta)}{r\,!} \text{ is } \frac{\alpha^r}{r\,!}$$

when θ is indefinitely diminished.

Now, the question arises, can we safely assume the above result without further proof, when r becomes indefinitely large? (For when n becomes indefinitely large, the number of terms and hence the number of factors in the numerator of the above expression and consequently r becomes indefinitely large). For a complete rigorous proof of the limiting value of the above expression, Art. 99 of *Hobson's Plane Trigonometry* can be consulted. Also see Appendix, Art. 7.

(4) sinh and cosh series.

$$\sinh x = x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \cdots\cdots$$

$$\cosh x = 1 + \frac{x^2}{2\,!} + \frac{x^4}{4\,!} + \frac{x^6}{6\,!} + \cdots$$

(i) When x is *real*,

for the sinh series,

$$Lt\left|\frac{u_{n+1}}{u_n}\right| = Lt\left|\frac{x^{2n+1}}{|\underline{2n+1}} \div \frac{x^{2n-1}}{|\underline{2n-1}}\right|$$

$$= Lt\frac{x^2}{2n(2n+1)} = 0, \text{ for all values of } x.$$

Hence, the series for sinh x is *absolutely convergent* for all values of x.

Similarly, it can be shown that the series for cosh x is *absolutely convergent* for all values of x.

(ii) When x is *complex*,

then also the series for sinh x and cosh x are *absolutely convergent*.

(5) Gregory's series.

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \cdots\cdots \text{ (} \theta \text{ being } \textit{real} \text{)}$$

$$Lt\left|\frac{u_{n+1}}{u_n}\right| = Lt\left|\frac{(-1)^n\tan^{2n+1}\theta}{2n+1} \div \frac{(-1)^{n-1}\tan^{2n-1}\theta}{2n-1}\right|$$

$$= Lt\frac{2n-1}{2n+1}\tan^2\theta$$

$$= \tan^2\theta.$$

Hence, the series is *convergent* for all values of θ for which $\tan^2\theta < 1$, *i.e.*, $-1 < \tan\theta < +1$,

$$\textit{i.e.,} \quad n\pi - \frac{\pi}{4} < \theta < n\pi + \frac{\pi}{4}.$$

The series is also *convergent* when $\tan^2\theta$ is equal to 1, *i.e.*, when $\tan\theta = \pm 1$, *i.e.*, when $\theta = n\pi \pm \dfrac{\pi}{4}$.

(6) sin θ and cos θ as infinite products.

We have, $\sin\theta = \theta\, \Pi\left(1 - \dfrac{\theta^2}{r^2\pi^2}\right)$ ... ... (1)

and $\cos\theta = \Pi\left[1 - \dfrac{4\theta^2}{(2r-1)^2\pi^2}\right]$ ... ... (2)

The infinite product (1) is absolutely convergent, since the series

$$\Sigma\left|-\frac{\theta^2}{r^2\pi^2}\right| \text{ or } \frac{\theta^2}{\pi^2}\cdot\Sigma\frac{1}{r^2} \text{ is convergent.}$$

The infinite product (2) is also absolutely convergent, since the series

$$\Sigma\left|-\frac{4\theta^2}{(2r-1)^2\pi^2}\right| \text{or } \frac{4\theta^2}{\pi^2}\cdot\Sigma\frac{1}{(2r-1)^2} \text{ is convergent.}$$

Each quadratic factor in the products (1) and (2) may be resolved into linear factors in θ; thus,

$$\sin\theta = \theta\left(1+\frac{\theta}{\pi}\right)\left(1-\frac{\theta}{\pi}\right)\left(1+\frac{\theta}{2\pi}\right)\left(1-\frac{\theta}{2\pi}\right) \quad \cdots \quad (3)$$

$$\cos\theta = \left(1+\frac{2\theta}{\pi}\right)\left(1-\frac{2\theta}{\pi}\right)\left(1+\frac{2\theta}{3\pi}\right)\left(1-\frac{2\theta}{3\pi}\right) \quad \cdots \quad (4)$$

The products in the forms (3) and (4) are *semi-convergent* because the series $\Sigma\,\dfrac{1}{r}$ and $\Sigma\,\dfrac{1}{2r-1}$ are divergent.

(7) sinh x and cosh x as infinite products.

Since, $\sinh x = x\, \Pi\left(1 + \dfrac{x^2}{r^2\pi^2}\right)$

and $\cosh x = \Pi\left[1 + \dfrac{4x^2}{(2r-1)^2\pi^2}\right]$

it follows from the previous paragraph that the infinite products for $\sinh x$ and $\cosh x$ are absolutely convergent.

7. An alternative proof for the expansion of cos θ and sin θ.

By writing θ/m and m in place of θ and n in (1) of Art. 51, we get after a little re-arrangement

$$\cos\theta = \cos^m\frac{\theta}{m}\left\{1 - \frac{(1-1/m)}{2\,!}\theta^2\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right)^2\right.$$
$$\left. + \frac{(1-1/m)(1-2/m)(1-3/m)}{4\,!}\theta^4\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right)^4 - \cdots\right\} \quad (1)$$

$$= \cos^m\frac{\theta}{m}\left\{1 - u_2 + u_4 - \cdots\cdots\right\} \text{ say.} \quad \cdots \quad (2)$$

Hence, so long as r is finite,

$$\text{we have } \underset{m\to\infty}{Lt}\frac{u_{2r+2}}{u_{2r}} = \frac{\theta^2}{(2r+1)(2r+2)}. \quad \cdots \quad (3)$$

If, therefore, we take $2r + 1 > \theta$, [strictly speaking if $\theta < \sqrt{\{(2r+1)(2r+2)\}}$], then by taking m (which is a positive integer here) large enough, we can always secure that on and after the term u_{2r}, the numerical value of the convergency-ratio of the series (1) shall be less than 1.

Hence, it follows that if $2r + 1 > \theta$, and m be only taken large enough, the value of cos θ will lie between

$$\cos^m\frac{\theta}{m}\{1 - u_2 + u_4 - \cdots + (-1)^r u_{2r}\} \quad \cdots \quad (4)$$

and $\cos^m\dfrac{\theta}{m}\{1 - u_2 + u_4 - \cdots + (-1)^r u_{2r} + (-1)^{r+1} u_{2r+2}\}.$

$$\cdots \quad (5)$$

Therefore, cos θ will always lie between the limits of (4) and (5) as $m \to \infty$.

Now, $Lt \cos^m(\theta/m) = 1$ [See *Art. 117*] ; $Lt\, u_2 = \theta^2/2\,!$; $Lt\, u_4 = \theta^4/4\,! \cdots\cdots Lt\, u_{2r} = \theta^{2r}/(2r)\,!$

$$Lt\, u_{2r+2} = \theta^{2r+2}/(2r+2)\,!$$

Thus, $\cos\theta$ lies between

$$1-\frac{\theta^2}{2\,!}+\frac{\theta^4}{4\,!}-\cdots\cdots+(-1)^r\frac{\theta^{2r}}{(2r)\,!} \quad \cdots \quad \cdots \quad (6)$$

and $$1-\frac{\theta^2}{2\,!}+\frac{\theta^4}{4\,!}-\cdots\cdots+(-1)^r\frac{\theta^{2r}}{(2r)\,!}+(-1)^{r+1}\frac{\theta^{2r+2}}{(2r+2)\,!}. \quad (7)$$

In other words, provided $2r+1>\theta$,

$$\cos\theta=1-\frac{\theta^2}{2\,!}+\frac{\theta^4}{4\,!}-\cdots\cdots+(-1)^r\frac{\theta^{2r}}{(2r)\,!}+(-1)^{r+1}R,$$

where $R<\theta^{2r+2}/(2r+2)\,!$.

Since, $\underset{r\to\infty}{Lt}\dfrac{\theta^{2r+2}}{(2r+2)\,!}=0$, we get ultimately

$$\cos\theta=1-\frac{\theta^2}{2\,!}+\frac{\theta^4}{4\,!}-\cdots\cdots \text{ to } \infty.$$

Similarly, the expansion of $\sin\theta$ can be obtained.

Note. That $\cos\theta$ lies between the series (6) and (7) can also easily be shown by considering the signs of the successive derived functions [See *sum no. 20 (a), Ex. IV (c), Authors' Differential Calculus.*]

CALCUTTA UNIVERSITY

MATHEMATICS—Honours

1. (*a*) Let $z=x+iy$, where x and y are real numbers and $i=\sqrt{-1}$.

(i) If the ratio $(z-i)/(z-1)$ is purely imaginary, show that the point z in the Argand diagram lies on the circle whose centre is the point $\frac{1}{2}(1+i)$ and whose radius is $\frac{1}{\sqrt{2}}$.

(ii) Find the roots of $z^n=(z+1)^n$, and show that the points which represent them in the Argand diagram are collinear.

(*b*) Show that $\tan(\alpha+i\beta)=\dfrac{2\sin 2\alpha+e^{2\beta}-e^{-2\beta}}{2\cos 2\alpha+e^{2\beta}+e^{-2\beta}}$.

1. Explain De Moivre's theorem.

(*a*) Prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$

$$=\cos(\tfrac{1}{2}n\pi-n\theta)+i\sin(\tfrac{1}{2}n\pi-n\theta)$$

(*b*) If $x+iy=c\cos(u+iV)$, prove that $u=$constant represents a family of confocal hyperbolas and $V=$const. represents a family of confocal ellipses.

1. Define the function Exp. z when z is a complex variable and deduce the addition theorem.

$$\text{Exp. } z_1+\text{Exp. } z_2=\text{Exp. }(z_1+z_2).$$

Find the sum of the series

(i) $\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta+\cdots$

(ii) $\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\cdots$

1. (*a*) Prove that the modulus of the product of two complex numbers is equal to the product of their moduli.

(*b*) Let z_1, z_2, z_3 be three complex numbers, such that

$$|z_1| = |z_2| = |z_3| = 1 \text{ and } z_1 + z_2 + z_3 = 0$$

Show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with centre at the origin.

(*c*) If $x + iy = \tan(u + iv)$ where $i = \sqrt{-1}$ and x, y, u, v are all real, show that the curves $u =$ const. represent a family of coaxal circles passing through the points $(0, \pm 1)$ and the curves $v =$ const. represent a system of circles cutting the first system orthogonally.

(Revised Regulation)

1. (*a*) When n is an odd positive integer, prove that

$$x^n + 1 = (x+1) \prod_{r=0}^{r=\frac{1}{2}(n-3)} \left\{ x^2 - 2x \cos \frac{(2r+1)\pi}{n} + 1 \right\}$$

(*b*) Solve the equation

$$(1+z)^n = (1-z)^n.$$